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GRAPH THEORY -- A MANAGEMENT TOOL FOR THE U. S. ARMY.(U)

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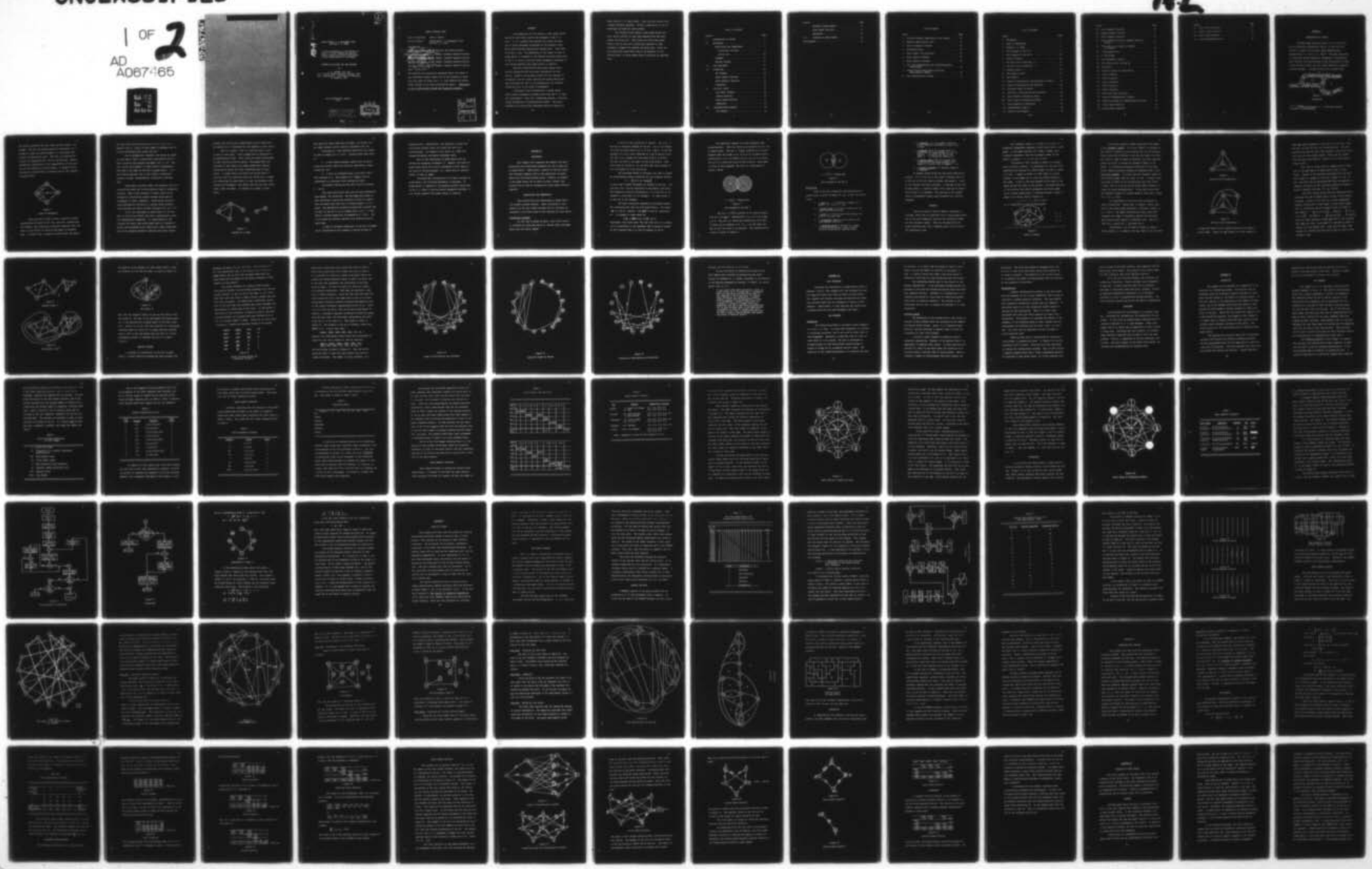

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GRAPH THEORY -- A MANAGEMENT TOOL
FOR THE U. S. ARMY

A thesis presented to the Faculty of the U. S. Army Command and General Staff College in partial fulfillment of the requirements of the degree

MASTER OF MILITARY ART AND SCIENCE

by

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Date: 28 May 1971

The opinions and conclusions expressed herein are those of the individual student author and do not necessarily represent the views of either the U. S. Army Command and General Staff College or any other governmental agency. (<u>References to this study should include the foregoing statement</u>.)

ABSTRACT

The proposition of this thesis is that graph theory should be more widely used by the managers of the U. S. Army. It is a powerful and flexible tool which has been and is being thoroughly researched at the abstract level. Many useful theorems dealing with graphs exist. They need to be put to use. The hypothesis of this paper is that if graph theory is compared to the methods currently being used by the U. S. Army in solving several managerial problems, it will become apparent that graph theory is superior.

Numerous authoritative texts about graph theory provide background and historical information for this thesis. However, the primary sources for new concepts of utilization are the recent journals and papers which have been published not only in the mathematical and research fields, but also in the field of management.

Following a brief introduction to graph theory, three typical managerial problems which face the U. S. Army are investigated. They are a scheduling problem, a facility layout problem, and a transportation problem. The three problems are solved using techniques which are typical of

those used by U. S. Army staffs. They are then solved using a graph theoretic approach. Finally, comparisons of the two techniques are made for each problem.

The finding of the study is that graph theory produces better results on some type problems than the techniques now being used. This study also finds that graph theory could be used as an additional approach to some problems to augment the methods now being used. There are also additional areas which need to be explored, at the working level, in which graph theory could play an important role.

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CHAPTER I

INTRODUCTION TO GRAPHS

Although graph theory may seem to be a relatively new and expanding mathematical theory, it was first discussed in a paper by the famous Swiss mathematician, Leonhard Euler in 1736 ((3, p. 188))*. The paper described a traditional problem among the townspeople of Königsberg. This Prussian city was situated on the intersection of three rivers and was divided into four areas which were connected by seven bridges. A sketch of the city is shown below:

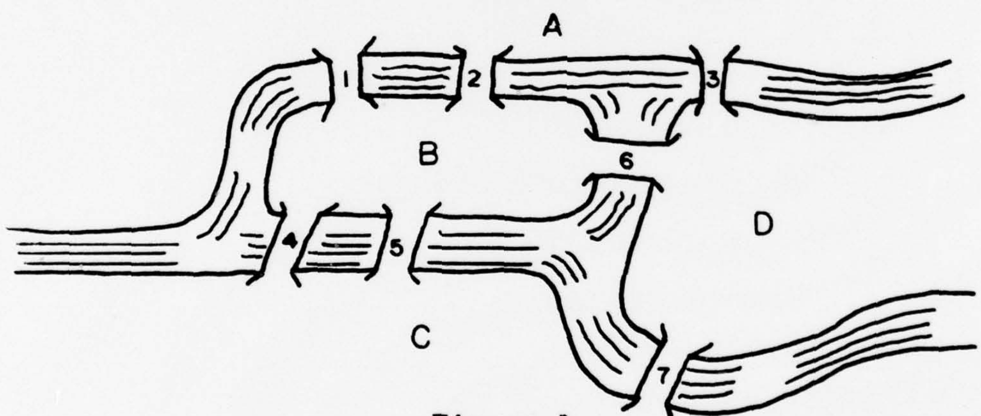


Figure 1

Königsberg

*Double parentheses (( )) indicate footnote reference to Bibliography.

The letters represent the four areas and the numbers, the bridges. The age old problem among the townspeople was to attempt to cross all seven bridges in a continuous walk without recrossing the route. This was, as everyone discovered, an impossible feat, but no one knew why. Euler's mathematical explanation of the impossibility of the problem was the beginning of graph theory. A synopsis of his discussion is given below to describe some of the elements of graph theory.

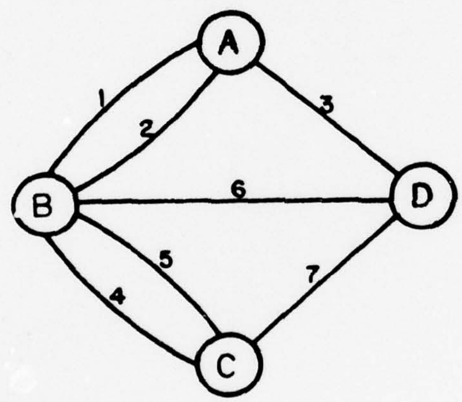


Figure 2

Graph of Königsberg

Euler pointed out that in such a network of nodes, representing the areas of the city, and arcs, representing the bridges, some retracing is necessary whenever there are three or more points at which an odd number of pathways meet. In other words, no matter at which point one begins,

one must enter each following point as many times as he departs from it. Hence, an even number of pathways must be at each succeeding point except the last.

From a mathematical viewpoint, the theory of graphs did not seem to offer a significant contribution since it dealt primarily with puzzles and games ((15, p. 3)). However, recent developments in other fields of mathematics have shed a new light on the uses of graph theory. It is now finding important uses in the fields of behavioral science, biology, transportation, electricity, and many more.

Graph theory provides simple, but powerful tools for constructing models and solving numerous types of managerial problems. Many of the problems in today's world can be analyzed by constructing complex systems using specific arrangement of their components. Graph theory provides a ready mathematical discipline to analyze these problems using theorems and theory already developed ((10, pp. 2-7)).

One of the advantages of graph theory is the fact that it can be used without the user's possessing an extensive mathematical background. An understanding of the theorems and theory upon which graph theory is founded requires some knowledge of set theory and a basic understanding of the concepts related to matrices and vector spaces.

However, the actual use of graph theory can be understood and applied as is illustrated by the examples in this thesis.

What is a graph? That is a question which occurs to many at this point. First, graphs will be explained in non-mathematical terms. Then, using set theory terminology, an exact definition will be given. The graphs which are discussed in this thesis are simple geometrical figures, consisting of points and lines connecting some of these points. In graph theory the points are called nodes or vertices, and the connecting lines are called arcs or edges. The terminology of graph theory is just now becoming standardized and throughout, this thesis will use the most common terms, nodes and edges. An example of a graph is shown below:

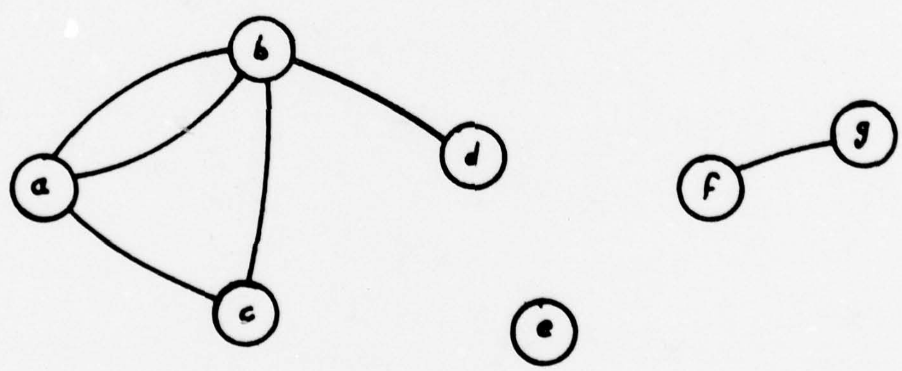


Figure 3

Example of a Graph

This graph has seven nodes and six edges. Of course, the two edges between a and b could be considered only one.

Following is a more technical definition of a graph as given by Berge ((2, p. 117)). A graph exists when there is:

1. A set N whose elements (nodes) will be represented by circles, which can stand for individuals, areas, situations, etc.

2. A set E of unordered pairs (a,b) with a and b both elements of N. The elements of E (edges) will be represented by continuous lines joining the nodes.

The graph G defined by the sets N and E is denoted G = (N,E).

The above definitions deal with the most fundamental type graphs, the undirected. Had the edges had arrows on them indicating a particular direction of flow or traffic, then the graph would have been a directed graph, or a digraph. The mathematical definition would have insisted that the pair in the set E, be an ordered pair, (a,b). In this thesis a directed graph will be denoted by D = (V,A). The V will stand for vertices and the A for directed arcs between them.

A need to introduce directions on the arcs of graphs can be illustrated by the problem of one-way streets or

communications. Additionally, the necessity to order the relationship between nodes can create the need for a directed graph. An example of the latter is a PERT (for Program Evaluation and Review Technique) chart.

More of the terminology of graph theory will be defined and explained in Chapter II. However, one field of graph theory which will not be considered in this thesis is the field of infinite graphs; i.e., those with an infinite number of nodes or edges.

Following this introduction to the basic concepts of graph theory, the following hypothesis is proposed: If graph theory is compared to the methods currently being used by the U. S. Army in solving several managerial problems, it will become apparent that graph theory is superior.

CHAPTER II

BACKGROUND

This chapter will enumerate and explain the basic definitions and terminology necessary for one to make use of graph theory. Additionally, several of the more important theorems commonly used in the application of graph theory will be stated without proof. Finally, an example of how graph theory can be used to solve a puzzle type problem will be used to introduce an actual graph theoretic approach.

DEFINITIONS AND TERMINOLOGY

Basic definitions and terminology in graph theory vary greatly between authors. Those introduced in this chapter are the most widely accepted and as a minimum are analogous to all those found in the research for this thesis.

Preliminary Concepts

At the risk of being too basic, this brief section is included for those who desire to refresh their knowledge about sets and vector spaces.

A set N is any collection of objects. Any object in the set is called an element of the set. If x is an element of N then we would write $x \in N$; if not, $x \notin N$. If all the elements of a set N were contained in a set M then we could say that N is a subset of M and write $N \subset M$, or $M \supset N$. Two sets are said to be equal if $N \subset M$ and $M \subset N$. This would then be written $M = N$, and mean that every element of M is an element of N and vice-versa.

One shorthand method of writing a set that is formed by establishing certain conditions for its elements follows:

$$N = \{n/n \text{ is an integer}\} \quad .$$

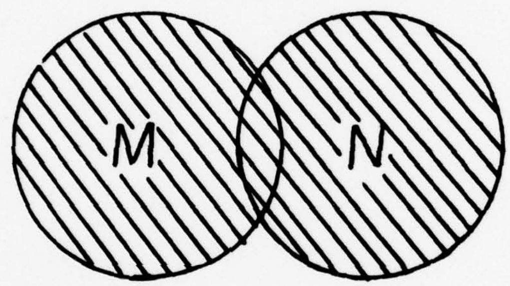
In this case n merely represents an element of the set. The vertical line / can be translated as the phrase 'such that'. The expression would be translated as, 'N is the set of all elements n, such that n is an integer'. In other words, N is the set of all integers.

Two other convenient mathematical shorthand symbols are associated with sets and with graph theory. The symbol $\Rightarrow$ is read as 'implies', and $\Leftrightarrow$ is read as 'equivalent to'. An example of these would be:

$$M \subset N \Leftrightarrow (m \in M \Rightarrow m \in N)$$

Translated, this would read 'The fact that set M is a subset of N is equivalent to the statement that m being an element of set M implies that m is also an element of set N'.

Two additional symbols are used frequently when discussing sets. Both will often be encountered by the user of graph theory. The set $P = M \bigcup N$ consists of all those elements that are either in M, or in N, or in both M and N. Symbolically, $m \in P$ if either $m \in M$, or $m \in N$, or both. The set P is called the union of sets M and N. Using what is known as Venn diagrams, the set P is illustrated in Figure 4 below.

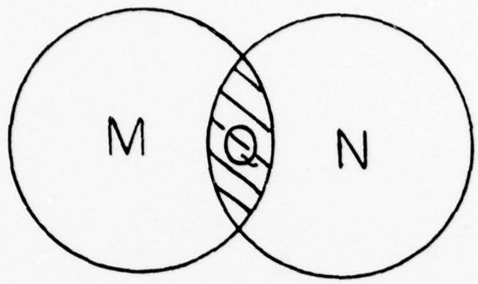


$P = M \bigcup N$ = Shaded Area

Figure 4

Venn Diagram of the Set P

The set $Q = M \bigcap N$ consists of all those elements that are in M <u>and</u> N. Symbolically, $m \in Q$ if $m \in M$ <u>and</u> $m \in N$. The set Q is called the intersection of M and N. If there are no elements in Q; i.e., Q is the empty set, then M and N are said to be disjoint. The intersection of M and N is shown in Figure 5.



$Q = M \cap N$ = Shaded Area

Figure 5

Venn Diagram of the Set Q

Definitions

Some of the more frequently used definitions of graph theory as given by Berge ((1, pp. 27-28)) are listed below:

1. A chain, μ , is sequence of edges in a graph, i.e. (a,b,c,····,p).
2. A simple chain is a chain in which all the edges used are different.
3. A cycle is a finite chain which begins and ends at the same node.
4. An elementary cycle is one in which each node appears only once.
5. A connected graph is defined as a graph in which there exists a chain joining every two arbitrarily selected nodes.

6. A component, C_a, of a graph is the set of all vertices connected to vertex a by a chain.

7. A subgraph (M,F) of a graph (N,E) is defined as the graph formed by a set of nodes M   N, and by the set of all edges F   E joining two nodes of M.

8. A partial graph (N,F) of a graph (N,E) is defined as the graph formed by all nodes in N and by a set of edges F   E.

9. A partial subgraph is a partial graph of a subgraph.

An example illustrating the last three definitions follows. If N is the set of all towns in Kansas, and if E is the set of all the roads in Kansas, the graph G = (N,E) is the complete road map of Kansas. A road map of only the primary roads is a partial graph, and a road map of Leavenworth County is a subgraph. The road map of the primary roads in Leavenworth County would therefore be a partial subgraph.

THEOREMS

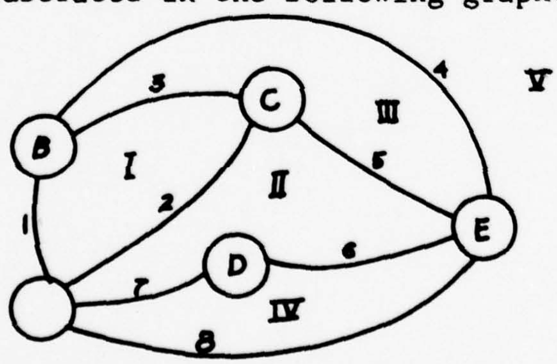
There are several special numbers, developed in theorems, which play an important role in using graph theory. Two of the most frequently used are the cyclomatic number and the chromatic number. They are introduced here by merely defining them, but a complete proof can be found in the references cited.

The cyclomatic number of a graph is the number of completely independent elementary cycles which exist in that graph. If the graph consists of n nodes, e edges, and p connected components, then the cyclomatic number of the graph, k, equals $e = n + p$ ((1, pp. 45-47)).

Correlated to this cyclomatic number is a formula known as Euler's Formula. To understand it, one must first understand the meaning of a planar graph, and what is meant by a face. A graph is planar if, when drawn on a plane surface, no two edges cut or cross one another except at the nodes. A face is the area bounded on all sides by the edges and has the nodes as the corners. One additional face is the infinite face which encompasses the entire graph. Given a planar graph which is connected and has n nodes, e edges, and f faces, Euler's Formula is:

$$n - e + f = 2. \quad ((2, p. 137)).$$

It is illustrated in the following graph:



$n = 5$

$e = 8$

$f = 5$

$n - e + f = 5 - 8 + 5 = 2 \qquad k = e - n + p = 8 - 5 + 1 = 4$

Figure 6

Euler's Formula

The second important number associated with graphs is the <u>chromatic number</u>. If a color (label) is assigned to each node of a graph in such a manner that any two adjacent nodes always have different colors and the number of different colors is made as small as possible, then this smallest number of colors is called the chromatic number of the graph. An interesting conjecture which has not yet been proven despite countless attempts by mathematicians is the <u>four color problem</u>. It states that the chromatic number of every planar graph, such as a map, is four. In other words, every map or globe can be colored using only four colors. In fact, this is true and can be easily shown; however, it cannot be proven mathematically. It has been proven that the chromatic number of a planar graph is less than or equal to 5 ((17, p. 112)).

An illustration of the four color conjecture is easily constructed. Assume that, in Figure 7 below, each node of the planar graph represents a different country, a, b, c, and d. The edges connecting the nodes would then represent the edges or boundaries between the countries. Each country would require a different color: Red for a, blue for b, yellow for c, and green for d.

Continuing, if as is shown in Figure 8, below, a fifth country, e, is added to the map, then it can only have

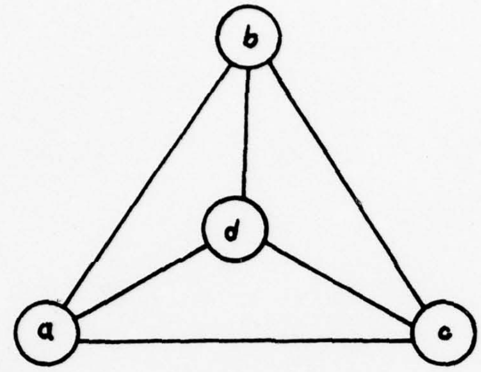


Figure 7

The Four Color Conjecture, 1

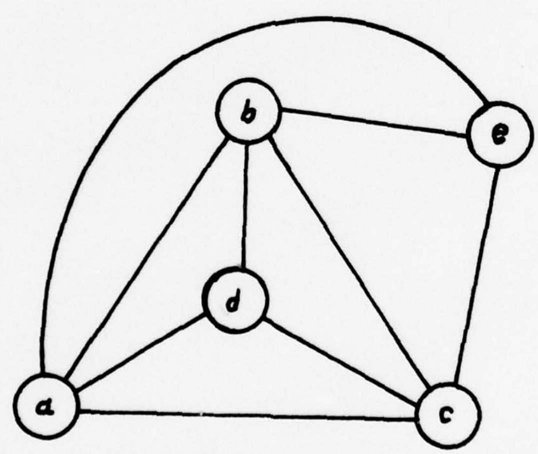


Figure 8

The Four Color Conjecture, 2

an edge with three of the original countries and remain a planar graph. Should an edge connect it to the country d,

that edge would necessarily cross one of the original edges and thus it would not be a two dimensional map. The same would be true had the node e been placed in any of the other four faces of the original graph. Thus e can be colored with the same color used for country d, since they have no common border.

This idea of using colors as indexes will be important to the application of graph theory later in this thesis. Several of the theorems associated with the chromatic index of a graph are outlined in Berge ((2, pp. 78-85)), and they will be referenced when they are applied. A complete discussion of the coloration problem of graphs is given in Busacker and Saaty ((4, pp. 82-90)).

One final concept which should be addressed before analyzing the uses of graph theory is that of the duality of graphs. Similar to the dual of a linear programming problem, there exists a dual graph to every planar graph. This is illustrated using the graph G given in Figure 9, with nodes N = $\{a,b,c,d,e,f,g\}$, edges E = $\{1,2,3,4,5,6,7,8,9\}$ and faces F = $\{f_1,f_2,f_3,f_4\}$. By joining every two faces sharing a common edge with a line crossing that edge only once, a new graph $\overline{G}$ is obtained as shown in Figure 10 in the dashed lines. Note that the edge 6 was crossed by a dashed line from face f_4 back to face f_4 thus causing a loop.

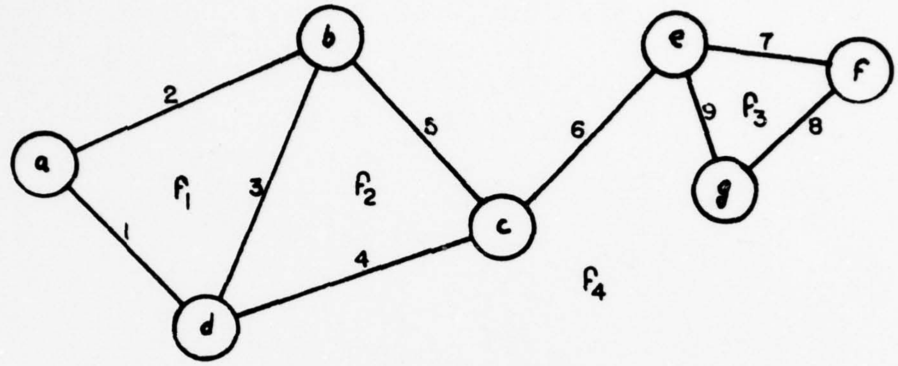


Figure 9

Example Graph, G

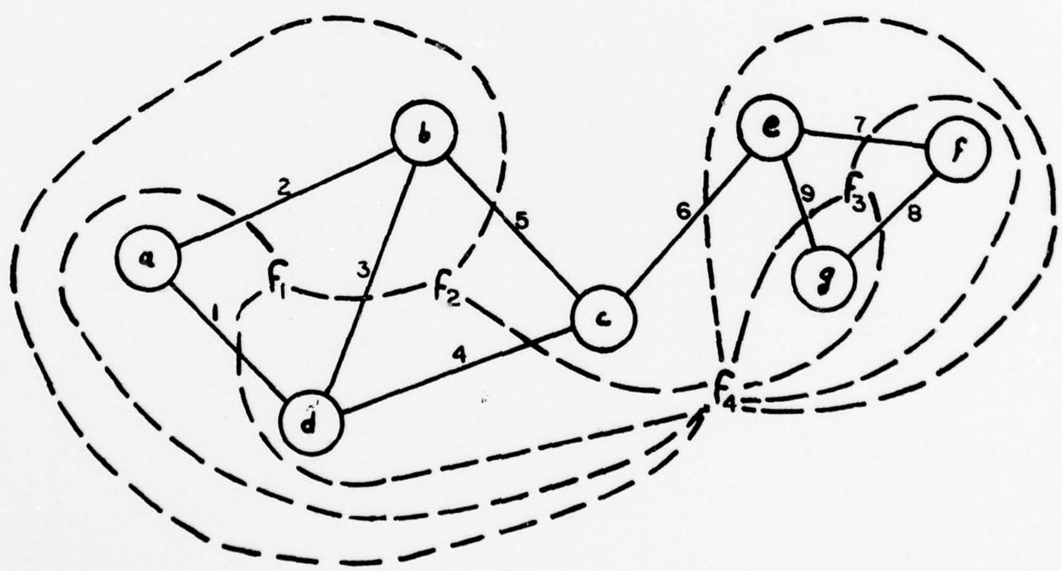


Figure 10

The Graphs G and $\overline{G}$

The graph $\overline{G}$ can be redrawn in a more direct manner, using the elements of F as the new nodes, as shown in Figure 11.

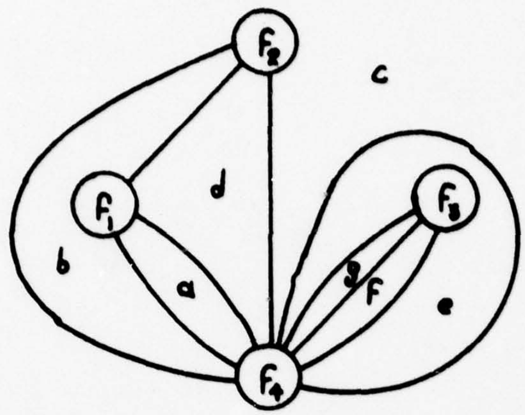


Figure 11

The Graph, $\overline{G}$

Note that the original nodes of G are now the faces of the dual graph $\overline{G}$. The uses of the dual graph will become apparent in the applications of graph theory in the later chapters. Suffice it to say, that the existence of a dual graph sometimes makes the solution to a graph theoretic problem easier, much in the same manner that the dual of a linear programming problem is sometimes the key to a simpler solution.

EXAMPLE PROBLEM

To present an introduction to the use of graph theory, a system transition problem has been borrowed from

Busacker and Saaty ((4, pp. 155-158)). This problem will not only demonstrate some of the methods used in applying graph theory, but it will also introduce additional concepts which will be helpful in the tests conducted in later chapters of this thesis.

The puzzle consists of a group of three rowing missionaries (M,M,M), two non-rowing cannibals (C,C) and one rowing cannibal, K, who are all on the bank, A, of a river. They must cross the river to bank B using a rowboat that can hold only two people. One additional, traditional restriction is the fact that the cannibals must never outnumber the missionaries on either bank of the river. If the system is restricted to the collection of people on bank A, the problem then becomes one of going from state (M,M,M,C,C,K) to (0) in a minimum number of steps. There are, as can be seen in Figure 12, twenty-four possible states of the system.

MMMCCK	MMCCK	MCCK	CCK
MMMCC	MMCC	MCC	CC
MMMCK	MMCK	MCK	CK
MMMC	MMC	MC	C
MMMK	MMK	MK	K
MMM	MM	M	0

Figure 12

System of Missionaries and Cannibals on Bank A

Those states underlined once violate the rules on bank A, while those underlined twice violate the rules on bank B. It is possible to construct a graph (non-planar) which represents the sixteen possible states on bank A as nodes, and at the same time represents the transitions of the boat trips as edges. As shown in Figure 13, there are twenty-five possible transitions. Each edge can represent a boat trip in either direction. Since the boat must move back and forth across the river, the edges must be used in an alternating manner between clockwise (representing departure from A) and counter-clockwise (representing arrival at A). After a bit of trial and error, this can be done using Figure 13. However, a systematic approach introduces an auxiliary graph with the same nodes, but with edges representing the round trips from A, to B, back to A. This graph is shown in Figure 14. The problem is now one of finding a chain from MMMCCK to 0. A chain does exist:

MMMCCK, MMMCK, MMMK, MMCK, MMCC, CCK, MC, 0

However, the intermediate states which were illustrated in Figure 13, but not in Figure 14, must be inserted.

MMMCCK, (MMCK), MMMCK, (MMM), MMMK, (MK),
MMCK, (MC), MMCC, (CC), CCK, (C), MC, 0

The entire path is shown in Figure 15. Thus the entire group was able to cross the river without the loss of a single missionary. The number of trips, thirteen, is a

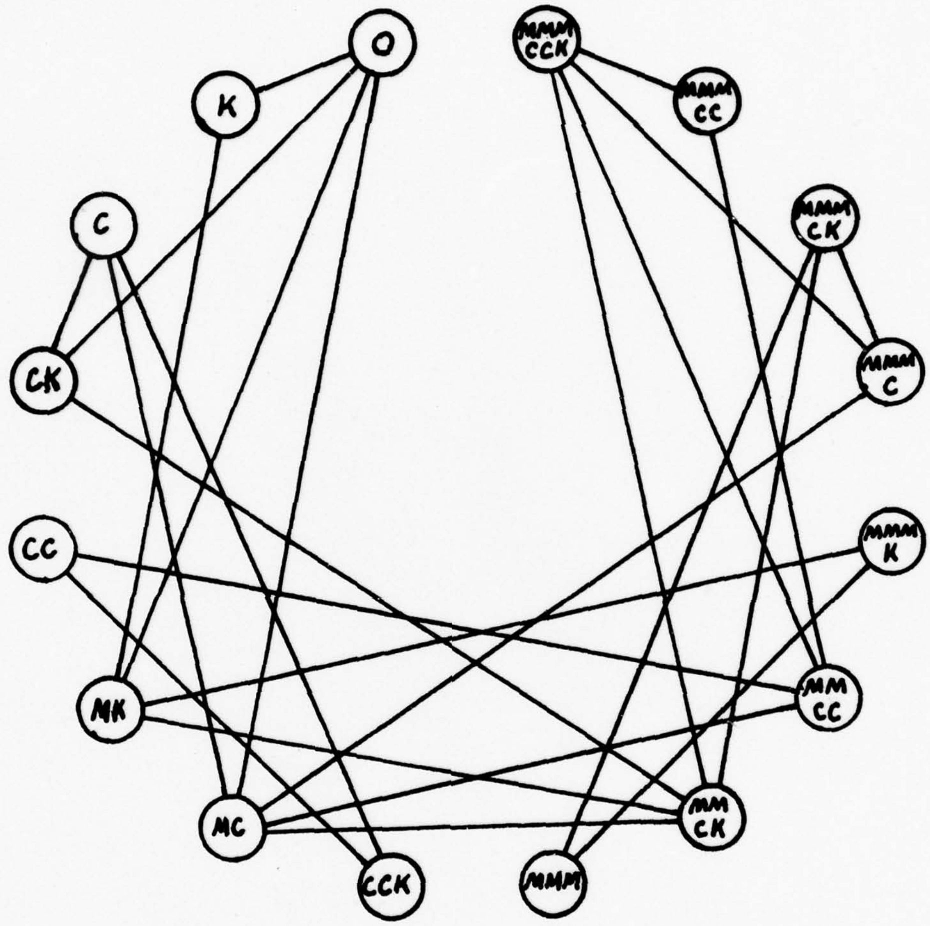


Figure 13

Graph of Missionaries and Cannibals

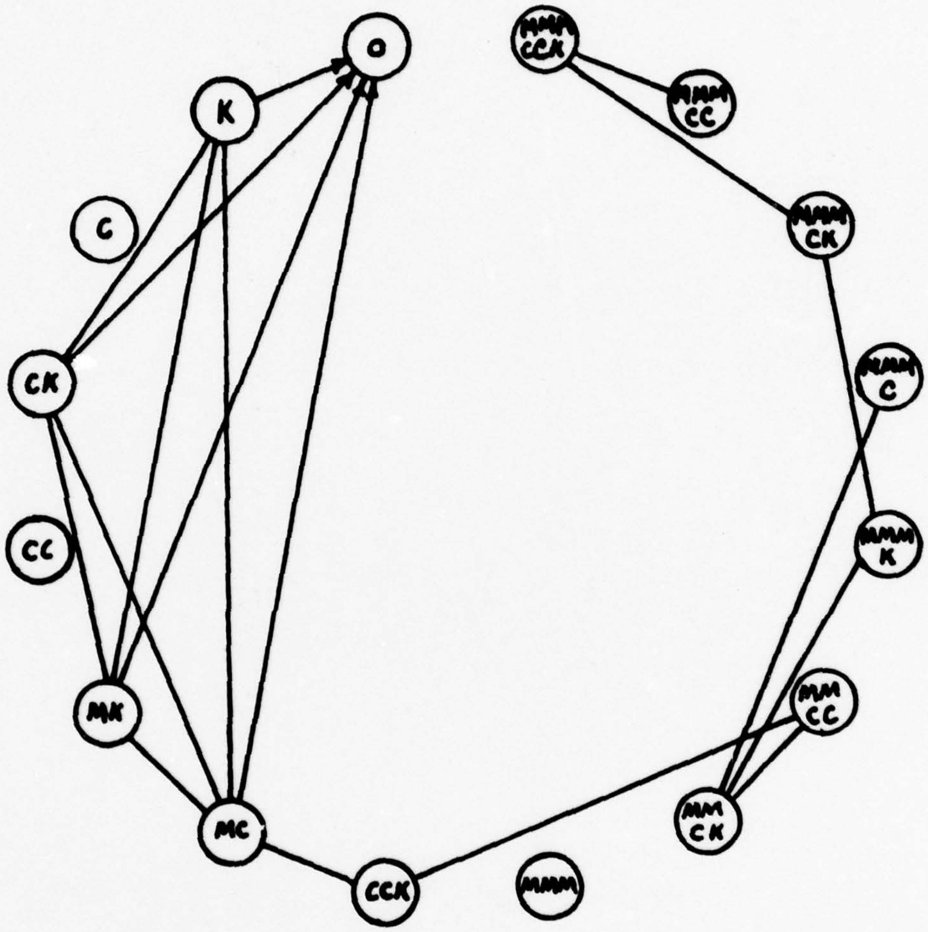


Figure 14

Auxiliary Graph of System

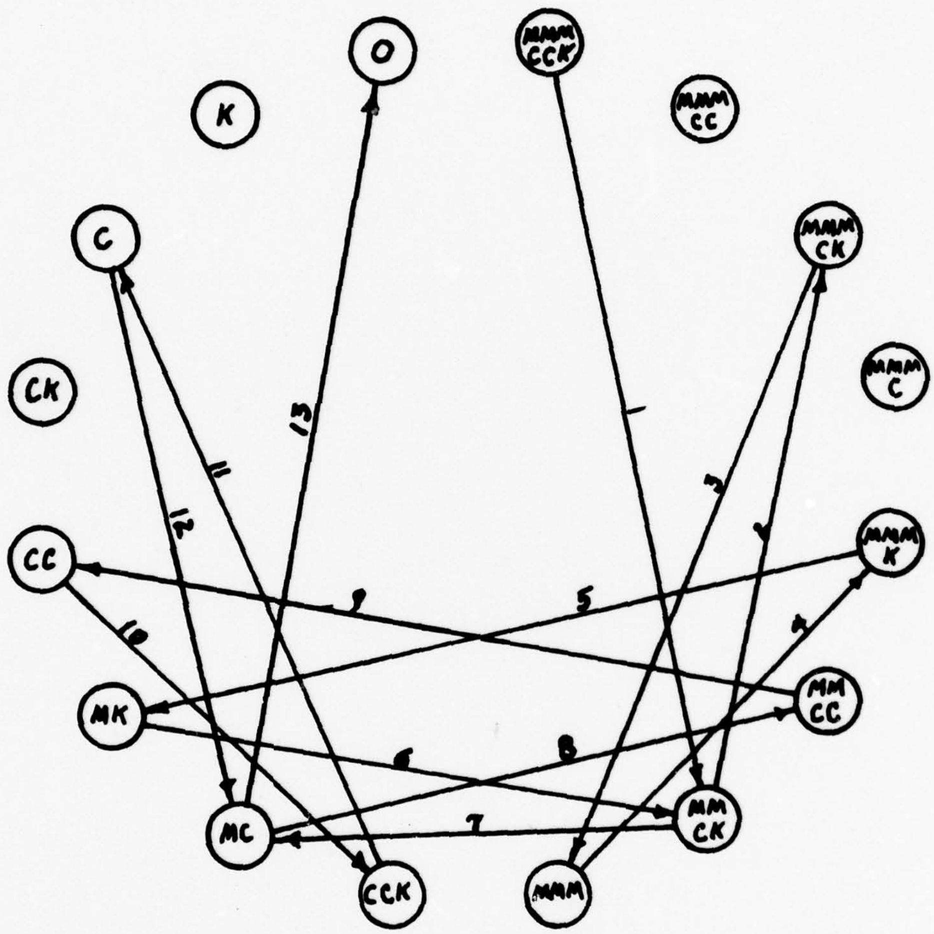


Figure 15

Solution to Missionaries and Cannibals

minimum, but the solution is not unique.

Having introduced and demonstrated graph theory this chapter will conclude by quoting from the Gibbs lecture of Professor R. L. Wilder, published in the Bulletin of the American Mathematical Society, 75 (1969), 19, and by Harris ((10, p. 1)).

> "I can recall that while teaching a course in graph theory 30 or 35 years ago, I recognized that far from being the dead field that it was regarded to be at the time, it offered great potential for research by a student without a great deal of background in classical mathematics. However, the recent resurgence of research in graph theory was apparently not due to any such considerations, but to the discovery that it had applications to problems in both the natural and social sciences--a fact that I suspected Cayley knew but had no time to pursue beyond some elementary work in chemical bonds."

CHAPTER III

TEST PROCEDURE

Following the introduction to graph theory given in Chapters I and II, this chapter will now introduce the test and comparison which will form the basis for this study. This chapter will briefly introduce and describe the three type problems which will be considered in depth in later chapters. Additionally, it will describe the overall test procedure which will be used throughout the thesis.

THE PROBLEMS

Scheduling

The scheduling problem is one which occurs frequently in the U. S. Army. It occurs most frequently in training or schooling situations, but it can occur at every level of Army management. Generally it takes the form of a plan or order which is to be issued. The plan is developed to arrange periods of time during which various groups of people or things are to accomplish specific tasks. The objective of the scheduling problem is to optimize the plan.

By optimize, it is meant that the number of tasks is maximized, or that the number of conflicts is minimized, or both. A conflict would occur when a particular group is directed to more than one task during the same time period.

The scheduling problem used in this thesis is completely hypothetical. It was developed through personal experience in the Army and through discussions which were held with numerous officers. Although it is hypothetical, it is realistic and typical of those type of scheduling problems which occur on numerous occasions. The setting is also fictitious, but it, too, is typical of countless U. S. Army situations.

Facility Layout

The background to the problem used in this thesis is typical of many problems which are presented by the Command and General Staff College. Again, it is completely hypothetical, and was developed in general terms to allow its correlation to real life problems.

A facility layout problem is primarily found in industrial engineering. However, in its general form it is applicable to numerous other fields. It deals with the production of a chart or map which shows the locations of activities within a facility that is being planned. There is normally a number of relationships that exist between the

activities. The activities should be arranged within the facility so that those which have mutual relationships are as close as possible. The objective of the layout problem is to maximize the number of relationships that are realized by the adjacency of activities.

Transportation

A general transportation problem is one which deals with the movement of one or more commodities from a group of locations to another group of locations. The type problem most frequently encountered involves only one commodity. A particular amount is at each origin, and another amount is required at each destination. There is a fixed "cost" associated with each path which connects an origin with a destination. The problem is to allocate the amounts of the commodity to the different paths in such a manner that the requirements are fulfilled and the total "cost" is a minimum. The "cost" may be expressed in terms of length or actual monetary costs.

There are many ways of formulating the transportation problem in mathematical terms. In Chapter VI one such formulation will be given. There are also many approaches to solving the transportation problem. Most of them involve a computer program which uses a linear programming approach. As discussed in more detail later, if a first feasible solu-

tion is close to the final solution, much computer time and hence money can be saved. The problem in this thesis deals with the finding of that first feasible solution.

As in the two previous problems, the situation and problem are fictitious. Similar problems occur frequently in the Army. On many occasions there are no computers available to the person faced with such a problem. Without an extensive mathematical background, it is extremely difficult to solve a transportation problem manually. The first feasible solution may be the one which is implemented.

PROCEDURE

Each problem will be discussed in a separate chapter. Following the introduction, the problem will be presented. It will first be solved using a technique which is currently available to U. S. Army managers. These techniques will be as modern and as analytical as those currently being used. Following the first solution, an original graph theoretic algorithm will be used to solve the same problem. Finally, a comparison of the two techniques will be made. The final chapter of the thesis will present conclusions and recommendations for further study.

CHAPTER IV

SCHEDULING

This chapter will be devoted to a comparison of two solutions to a scheduling problem. One solution will be developed using the techniques normally available to lower level military staffs or the staffs at training centers. It will rely heavily on the experience of the members of those staffs, and on the methods which have been passed on from persons who have worked on similar problems at those positions in the past. While this may create what appears to be a semi-analytical approach, the purpose of this chapter is to compare what is actually done with what could be accomplished using techniques outlined in this thesis. The second solution to the same problem will be obtained using graph theory.

The hypothetical situation which will be developed initially will portray the military scheduling problem. All the requirements which will be applicable to both solutions will be presented. The first approach will then solve the problem and present the solution. A graph theoretic

approach will then be used and its solution will be presented in the same format as the first. Finally a comparison of the two methods of solution will be made.

THE PROBLEM

Fort Swumpy is a U. S. Army basic training center located somewhere in the U. S. The staff of the training center has a normal complement of officers and sergeants. The scheduling branch of the G3 section consists of two officers and four sergeants, and is responsible for the scheduling of training for both the basic training companies and the lager units on the post. One of the officers, Major Lamod, has been given the task of scheduling the final week of basic training for the three regiments which are completing basic training. Unfortunately, due to a national emergency the last three weeks of training must be compressed into this one week of five training days. These recruits will be sent to active duty units for their Advanced Individual Training prior to deployment with those units.

The commanding general of Fort Swumpy, to expedite the training, has directed that as many companies as necessary be given a block of training simultaneously. He has also directed that no conflicts of scheduling occur. A conflict would exist if a particular company were scheduled

for two different periods of training on the same day. Each of the eight remaining subjects, based on mobilization standards, requires one complete day of training. To accomplish the mission in the best manner possible, the G3 has further directed that the minimum number of subjects be excluded from the minimum number of companies' training schedules. Table 1 shows the list of subjects which must be included in the nine companies' schedules for the five day training week. The subjects are arranged in order of their priority as directed by the G3. If a subject must be omitted from a companies' schedule, the larger the number the better.

Table 1

Training Subjects Remaining
at Fort Swumpy

I.	Quick fire range
II.	Preparation for overseas replacement qualification
III.	Hand grenade range
IV.	Close combat course
V.	Map reading and land navigation
VI.	Physical combat proficiency test
VII.	Bayonet course
VIII.	Gas chamber

Due to the staggered training schedules which the nine companies of the three regiments have followed, they are at various stages of completing the required course. Using the Roman numerals given in Table 1, Table 2 indicates those subjects which each company has not yet completed.

Table 2

Subjects Remaining per Unit

Unit Code	Company	Subjects	Total Days
1	A-1	I,III,IV,VI,VII	5
2	B-1	IV,V,VI,VIII	4
3	C-1	I,III,IV,VII,VIII	5
4	A-2	II,III,VI,VII	4
5	B-2	I,III,IV,VII,VIII	5
6	C-2	II,V,VI,VIII	4
7	A-3	I,IV,V,VIII	4
8	B-3	II,III,VI,VII,VIII	5
9	C-3	II,V,VI,VIII	4

It appears at first glance that, since each company has only five or less days remaining to complete, it would be possible to schedule all companies for all the training. However, due to manpower shortages at Fort Swumpy, it will

be possible to conduct each subject only once during this final week, which will be called "Crunch Week". The operation will be called "Operation Crunch".

MAJOR LAMOD'S SOLUTION

Initially, realizing that the solution to the scheduling problem must be based on the number of companies taking each test, Major Lamod constructs the chart shown in Table 3, below. This shows the unit codes originally given in Table 2.

Table 3

Units Arranged by Subject

Subject	Units	Total
I	1,3,5,7	4
II	4,6,8,9	4
III	1,3,4,5,8	5
IV	1,2,3,5,7	5
V	2,6,7,9	4
VI	1,2,4,6,8,9	6
VII	1,3,4,5,8	5
VIII	2,3,5,6,7,8,9	7

He then constructs a chart indicating the five days of training on the left and the eight subjects across the top. This chart is shown in Table 4 below.

Table 4

Scheduling Chart

Day \ Subject	I	II	III	IV	V	VI	VII	VIII
Monday								
Tuesday								
Wednesday								
Thursday								
Friday								

To arrive at an optimum solution to the scheduling problem, Major Lamod then constructs eight rectangular cards corresponding to the unit groupings in Table 3. One card, corresponding to subject III, looks like this, $\boxed{1,3,4,5,8}$. By placing the cards on the rectangles in the scheduling chart, Major Lamod attempts to minimize the number of units which must be deleted from the training. He realizes, of course, that there are 5^8 or 1,953,125 ways of arranging the 8 cards on the 40 rectangles. He decides to attempt only those which appear most promising.

He develops his own search algorithm in which the lower numbered, more important, subjects are given early in the week and the other cards are then moved from rectangle to rectangle in his attempt to minimize the deleted subjects. After he tries approximately one hundred different combinations, Major Lamod decides that the two schedules shown in Table 5 below are typical of the optimum solution. It can be seen that Major Lamod places the lower numbers in the schedule first and allows the deletions to occur in the higher numbered subjects. He then presents the two schedules to the G3 and suggests that the first be selected since it has the fewer number of subjects deleted from the companies' training. The schedule which Major Lamod recommends is reproduced below in Table 6 in a more readable style.

The G3 of the Fort Swumpy Training Center is a bit dubious of the schedule which Major Lamod has presented; therefore, he calls in the other officer from the scheduling division of the section and asks him to come up with a solution to the same problem.

MAJOR REKCOH'S SOLUTION

Major Rekcoh decides to analyze the problem using graph theory. It appears to him that the eight subjects could represent the nodes of a graph, and that the edges of

Table 5

Major Lamod's Two Solutions

Subj. / Day	I	II	III	IV	V	VI	VII	VIII	
Mon.	1,3,5,7	4,6,8,9							
Tues.			1,3,4,5,8		2,6,7,9				
Wed.				1,2,3,5,7		~~1~~, ~~2~~, 4 6, 8, 9			
Thur.							1,3,4,5,8		
Fri.								2, 3, 5, 6 7, 8, 9	
Miss. Subj.						1, 2			Total 2

Subj. / Day	I	II	III	IV	V	VI	VII	VIII	
Mon.	1,3,5,7	4,6,8,9							
Tues.			1,3,4,5,8		2,6,7,9				
Wed.				1,2,3,5,7					
Thur.						1, 2, 4 6, 8, 9			
Fri.							1,3,4,5,8	2, ~~3~~, ~~5~~, 6 7, ~~8~~, 9	
Miss. Subj.								3,5,8	Total 3

Table 6

Major Lamod's Schedule

Day	Subject	Companies to Attend
Monday	I, Quick Fire Range	A-1, C-1, B-2, A-3
	II, POR	A-2, C-2, B-3, C-3
Tuesday	III, Hand Grenade	A-1, C-1, A-2, B-2, B-3
	V, Map Reading	B-1, C-2, A-3, C-3
Wednesday	IV, Close Combat	A-1, B-1, C-1, B-2, A-3
	VI, P.C.P.T.	A-2, C-2, B-3, C-3
Thursday	VII, Bayonet	A-1, C-1, A-2, B-2, B-3
Friday	VIII, Gas Chamber	B-1, C-1, B-2, C-2 A-3, B-3, C-3

Note: Companies A-1 and B-1 will delete P.C.P.T.

the graph could represent the possible conflicts in which one or more companies could be scheduled for the same subject on the same day. Using the data given in Table 2, he constructs the graph in Figure 16.

Major Rekcoh's graph is constructed in the following manner: The nodes represent the subjects of instruction, I through VIII. The edges represent the fact that one or more companies must attend the subjects represented at either end of the edge. The small numbers on the edges represent the unit codes of the companies needing the subject at either end. The total number of companies on any edge is the weight on that edge. The small arabic number in the top of the nodes represent the number of edges which are incident with that node, and referred to herein as the flux. The small arabic numbers at the bottom of the node represent the total number of companies contained on the edges which are incident on that node.

Having constructed the graph based on the data presented, Major Rekcoh used the following algorithm to determine an optimum solution. If the chromatic number of the graph is five or less, each color can represent one day of "Crunch Week" and the subject nodes of the graph which can be "labeled" by a particular color can be scheduled on those days. He begins by selecting five colors; red, blue, green,

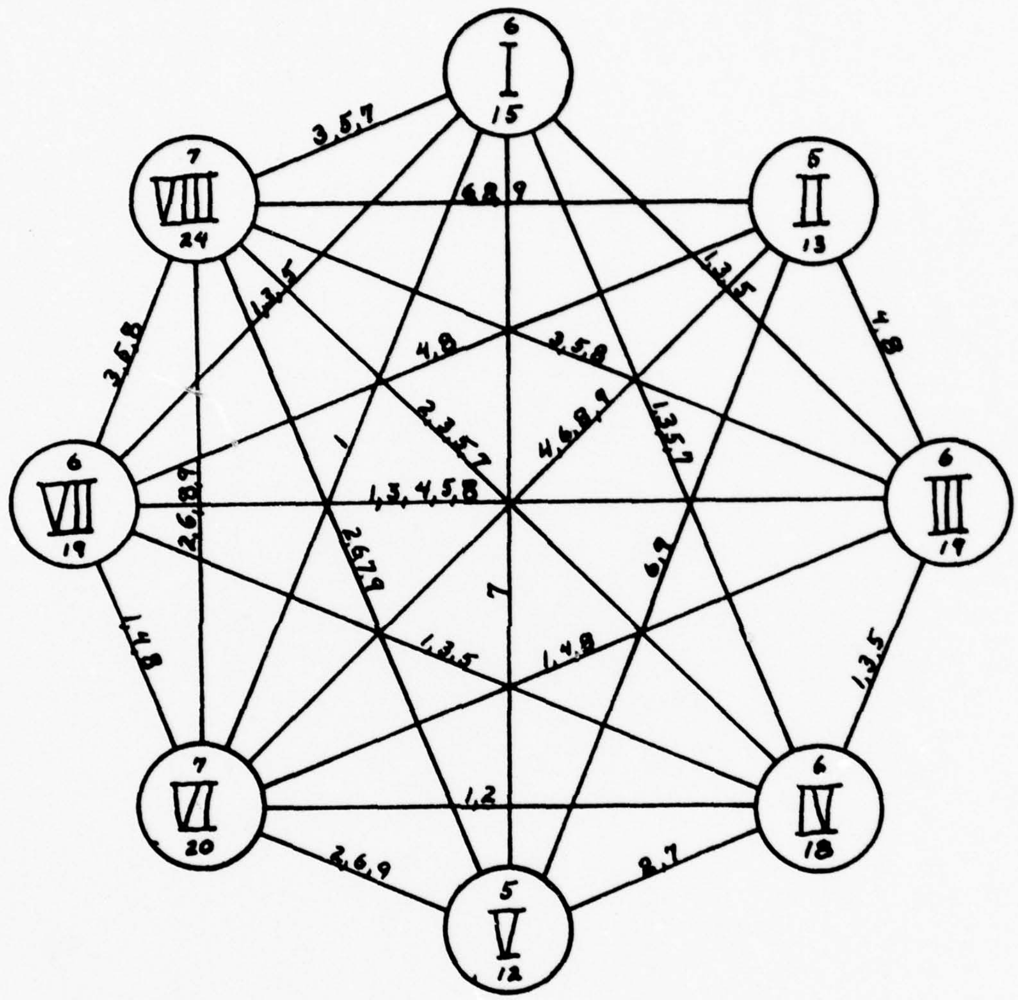


Figure 16

Major Rekcoh's Scheduling Graph

yellow and orange. He then begins the "labeling" by coloring the most connected nodes first, using the numbers in the top of nodes. In case of ties the selection process then reverts to the numbers in the bottom of the nodes. Thus, node VIII is first colored red, followed by node VI which is colored blue. A tie between both the upper and lower numbers in nodes VII and III allows Major Rekcoh to create one more rule of selection. In case of such a tie, the more important node is colored first. In this case, node III becomes green and node VII, yellow. According to the selection procedure node IV is colored orange.

Having used five colors, representing the five training days in "Crunch Week", Major Rekcoh now attempts to color the remaining nodes using the same five colors. His objective is to color each node using a color with which it is not connected. Therefore, node V can be colored yellow or green, and node II can be colored orange. Major Rekcoh selects green for V and orange for II. That leaves only I. It is not connected to II, but it is connected to IV, therefore, it cannot be colored orange. It is connected to nodes of all five colcrs. The algorithm now calls for one or more edges to be destroyed in an optimal fashion. That would mean omitting one of the subjects at one end of an edge for the companies on the edge. Major Rekcoh analyzes all the

edges which are incident with node I. He selects those with the fewest companies, least weight. Two edges represent only one company each. The I-VI edge has company 1 and the I-V edge has company 7. The optimal selection process then dictates that the I-VI edge be destroyed since subject V is more important than subject VI. In other words, company 1 will not take subject VI. Then node I can be colored blue. The final colored graph appears in Figure 16A. The chromatic number of this graph is five.

Major Rekcoh then constructs the training schedule for the nine companies using the graph shown above. Monday, represented by blue, is used for subject I and VI; Tuesday, orange, is used for subjects II and IV. Wednesday, green, is used for III and V. Thursday, yellow, is used for VII. Friday, red, is used for VIII. The final schedule is shown in Table 7. Only one company, A-1, must miss only one subject, VI.

COMPARISON

It is obvious that Major Rekcoh's graph theoretic solution produced a better solution to the scheduling problem. However, it is just as obvious that, given enough time, Major Lamod's approach would have produced the same solution. The advantage of using a graph in the solution

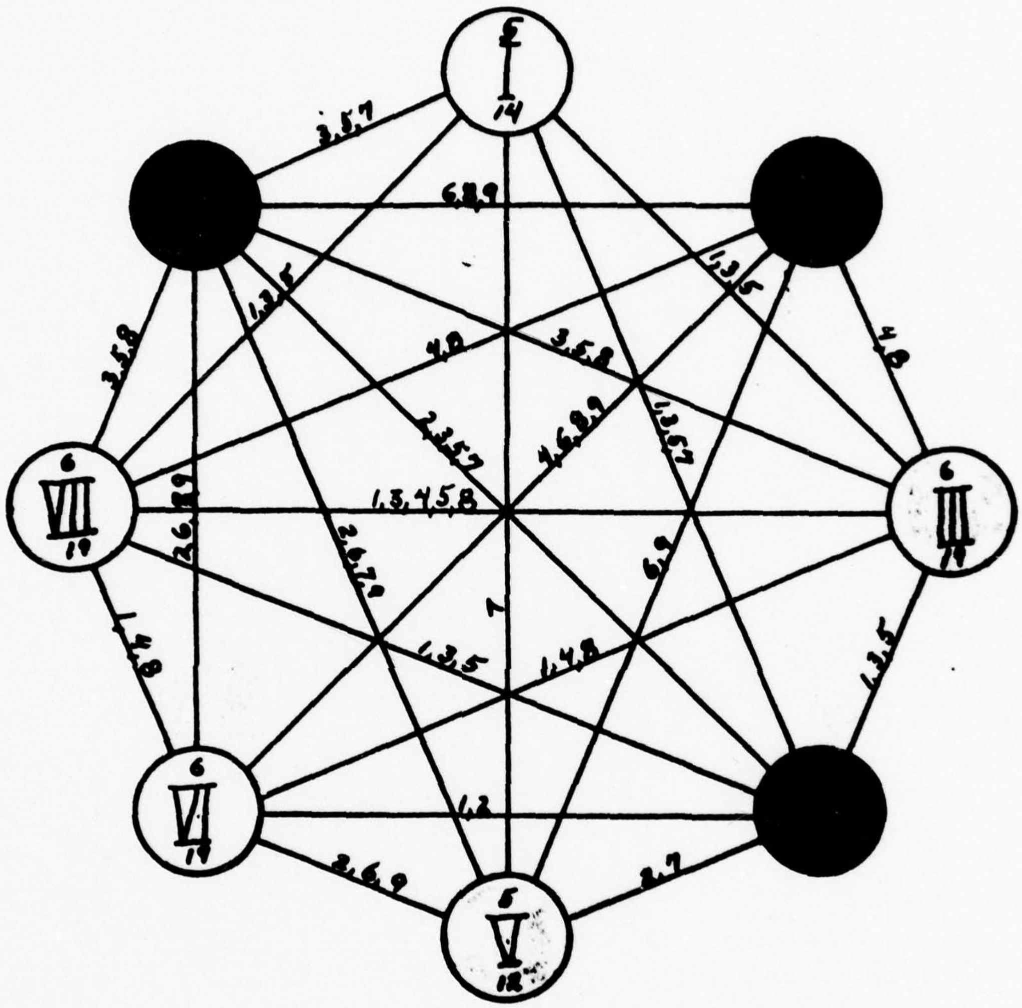


Figure 16A

Final Graph of Scheduling Problem

Table 7

Major Rekcoh's Schedule

Unit Codes	Companies	Subject	Day	Color
1,3,5,7	A-1,C-1,B-2,A-3	I	Mon.	Blue
2,4,6,8,9	B-1,A-2,C-2,B-3,C-3	VI	Mon.	
4,6,8,9	A-2,C-2,B-3,C-3	II	Tues.	[redacted]
1,2,3,5,7	A-1,B-1,C-1,B-2,A-3	IV	Tues.	
1,3,4,5,8	A-1,C-1,A-2,B-2,B-3	III	Wed.	Green
2,6,7,9	B-2,C-2,A-3,C-3	V	Wed.	
1,3,4,5,8	A-1,C-1,A-2,B-2,B-3	VII	Thur.	Yellow
2,3,5,6 7,8,9	B-1,C-1,B-2,C-2 A-3,B-3,C-3	VIII	Fri.	[redacted]

of a scheduling problem of this type is the fact that the optimal solution is guaranteed as the initial solution.

An argument against comparing graph theory with Major Lamod's approach might be the fact that, given a computer, he could have tested all 1,953,125 possibilities. The computer could also have been used with a program developed from Major Rekcoh's selection algorithm. In fact, the program flow chart can be drawn directly from the description given. See Figure 17. This second program would not have to search the entire set of possible solutions, but would zero in on the one best solution. Programs of this type are generally accepted as being more efficient, particularly when the solution sets are extremely large.

It is also possible to establish upper and lower bounds on the chromatic number of a graph. The following theorem is given by Busacker and Saaty ((4, pp. 79-80)). To establish the theorem on bounds, they first introduced the concept of the complementary graph. It is obtained by deleting from a complete graph with n nodes, all those edges contained in the original graph. The complementary graph, G', of the graph G which was used in the scheduling problem, is shown in Figure 18.

The theorem from Busacker and Saaty states that if k and k' are the chromatic numbers of a graph G with n nodes

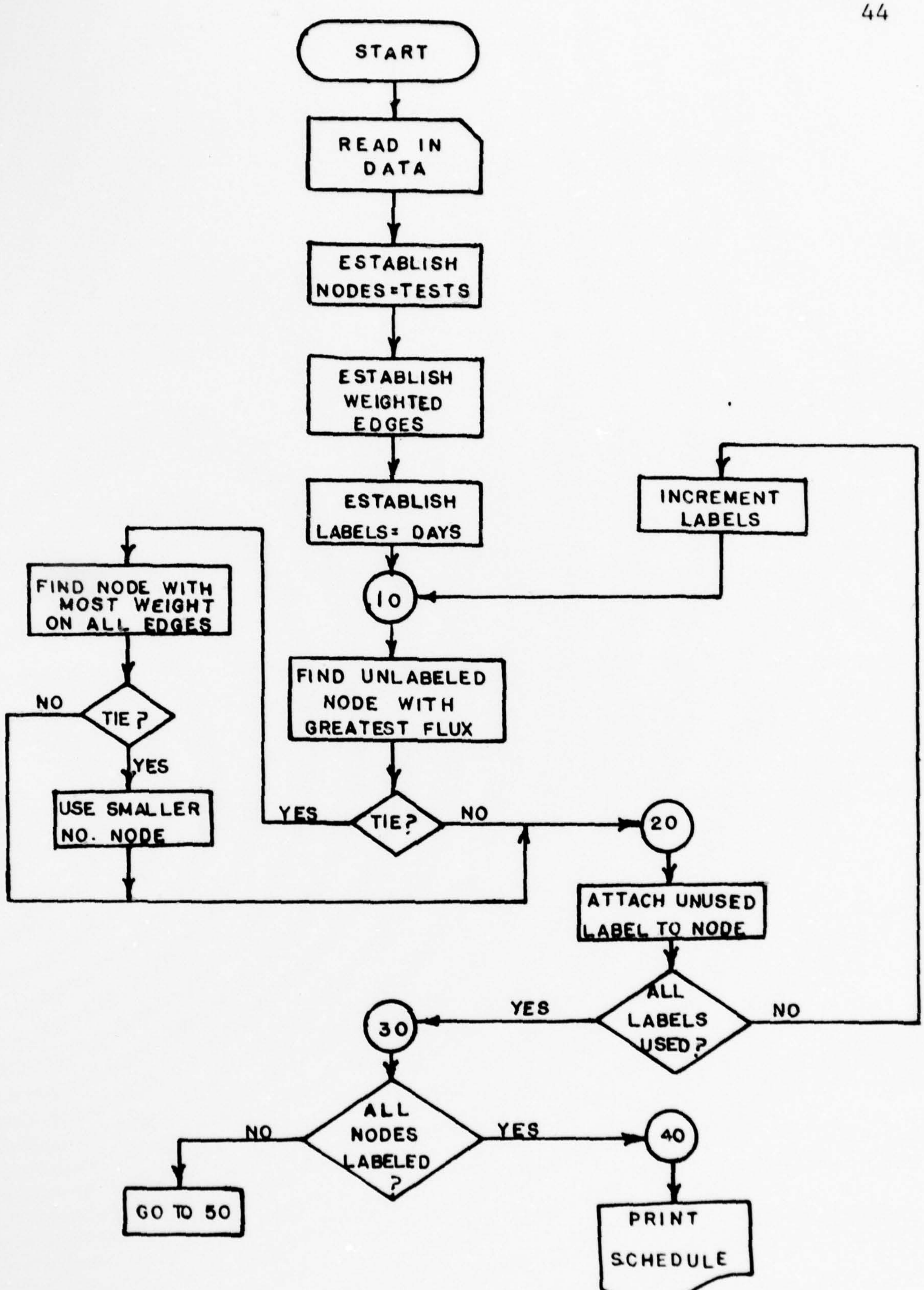


Figure 17

Flow Diagram for Scheduling

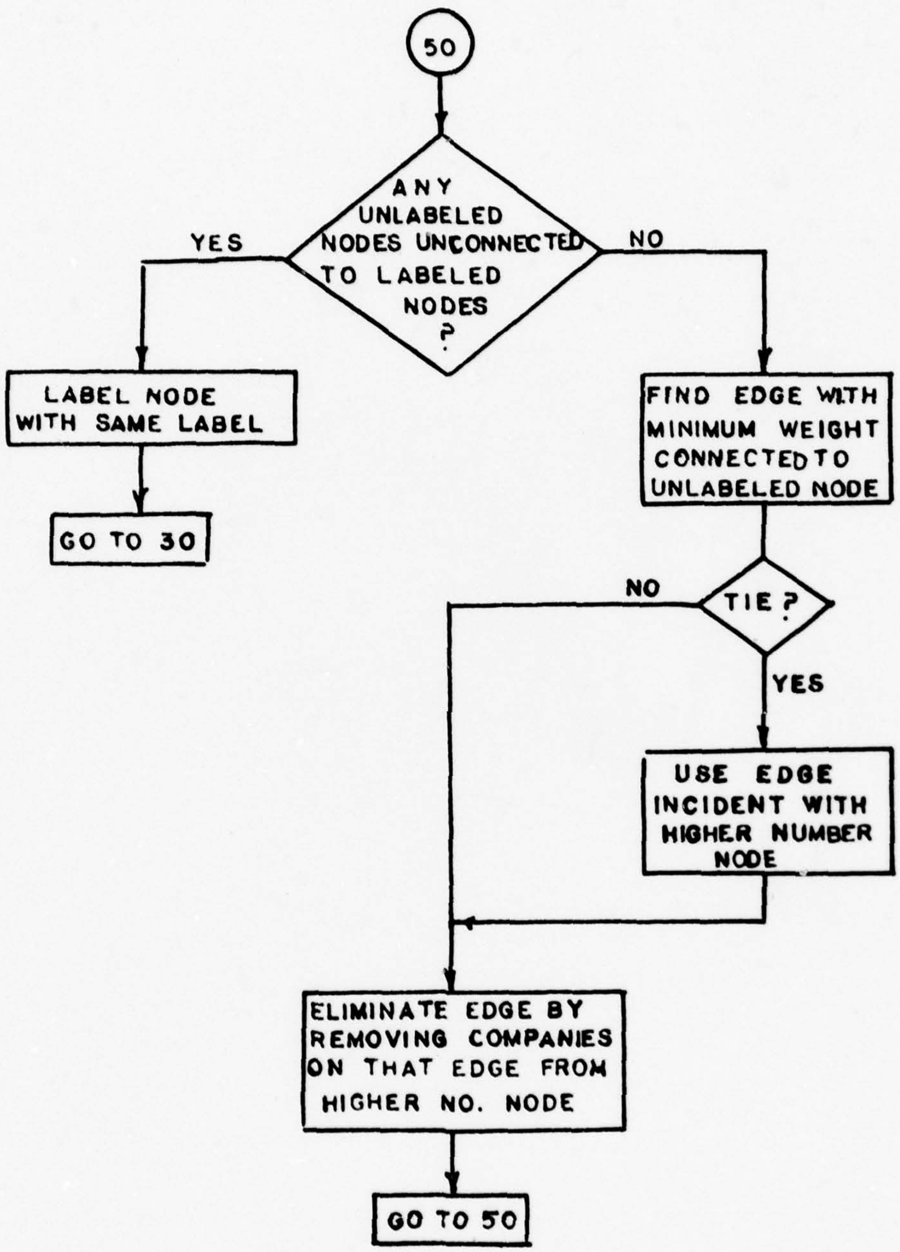


Figure 17

(Continued)

and its complementary graph G', respectively, then

$$2\sqrt{n} \leqslant k + k' \leqslant n + 1$$

$$\text{and} \quad n \leqslant kk' \leqslant \left(\frac{n+1}{2}\right)^2 .$$

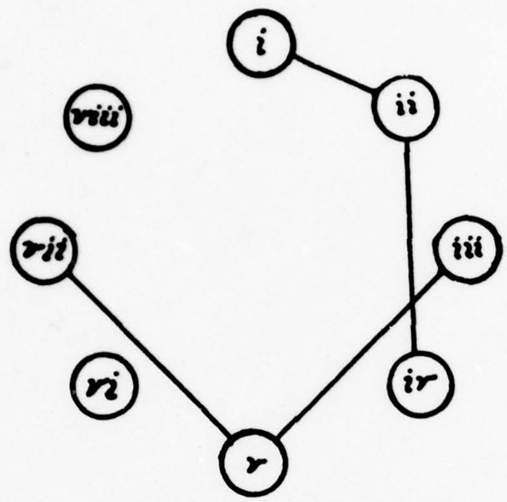


Figure 18

Complementary Graph, G'

In the scheduling problem above with graphs G, Figure 15 and G', Figure 18, this theorem would have told Major Rekcoh some facts about his problem. The chromatic number k', for G' is obviously 2. That is, all nodes could be colored using two colors without any two connected nodes being the same color. That results in the following:

$$2\sqrt{8} \leqslant k + 2 \leqslant 8 + 1 = 9$$

$$\text{and} \quad 8 \leqslant 2k \leqslant \frac{81}{4} = 20.25$$

$$\text{or} \quad 5.6 \leqslant k + 2 \leqslant 9$$

$$\text{and} \quad 4 \leqslant k \leqslant 20.25$$

$$\text{or} \quad 3.6 \leq k \leq 7$$
$$\text{and} \quad 4 \leq k \leq 20.25$$

Using the closer bounds of the two inequalities would have told Major Rekcoh that:

$$4 \leq k \leq 7$$

This would have given him a range of values in which the number of days required for scheduling would lie. On larger problems these bounds would be most helpful and an analysis of the complementary graph would provide insight into the best method to use in selecting the edges for destruction.

This graph theoretic solution to a general scheduling problem can be employed without reference to other mathematical disciplines. It is practical, in that it can be employed even by the scheduler sitting at the range on a tree stump. All he needs is paper and pencil. The general characteristics of these type problems are a set of time periods (nodes) during which a set of elements (companies, people, events, etc.) are supposed to perform or attain a certain set of tasks or states (tests, classes, polarity, size, etc.). Using the problem given in this chapter as an example, a U. S. Army manager who is asked to schedule any activity involving those three sets of parameters could use graph theory and obtain an optimal solution.

CHAPTER V

FACILITY LAYOUT

This chapter will deal with the problem of locating interrelated activities within a facility that is being planned. This problem can be related to many different situations. The activities can be offices, warehouses, classrooms, or any other type functionalized grouping. The problem, which will be used for the comparison test, will be completely hypothetical. It will concentrate on a U. S. Army base development plan, and discuss only the aspect of laying out the block plan for the various internal groupings. The basic problem will first be introduced. All of the constraints and requirements will be listed and the problem will be formulated in terms of what form the final layout should take.

The layout problem will then be solved using a rather sophisticated computer technique which was developed by Major Robert C. Lee in his MS Thesis ((12)). It has also been discussed in The Journal of Industrial Engineering ((13)) under the title CORELAP--COmputerized RElationship LAyout Planning. Major Lee also presented his technique

before a meeting of the Society of Industrial Engineers in 1970. As indicated by its title, CORELAP relies on the use of a computer. Therefore, to make a valid comparison, the second solution to the same problem using graph theory will also rely on the use of a computer. The actual programs will not be presented, only the formulation of the problem, the flow diagrams and the solutions. Following the graph theory solution, a comparison of the two methods will be made.

THE LAYOUT PROBLEM

The U. S. Army is considering a contingency plan to deploy a ground force into a hypothetical country, LOGANIA. The unified command headquarters has directed that the base development plan be produced and published as part of the contingency plan. One portion of the base development plan is the proposed layout of the particular areas within the different installations. One of the largest combat service support installations which is being planned for LOGANIA is a maintenance and supply depot. It is to be located outside the main port city and will be constructed following deployment of combat forces.

The area has been agreed upon by the LOGANIAN government and the unified headquarters. It is a relatively

flat area that will accomodate the entire complex. There are no advantages to being located in any particular part of the area. There are sixteen subordinate units which are to be located in the complex, and they conduct interdependent activities. For the sake of brevity and clarity they will be referred to as Units 1 through 16 with no attempt to list the type units. The sixteen units, their area requirements and the interrelationship requirements are listed in Table 8. The column on the right indicates a total closeness rating which will play an important part in the CORELAP solution. Note also, that the matrix is symmetric and repeats itself below the main diagonal.

This chart was developed through the combined efforts of the sixteen units and the unified staff. It is somewhat simplified for this discussion. An "undesirable" closeness rating could be included as a negative number. A triangular relationship chart which includes all of these characteristics was designed by Richard Mather & Associates ((14)) and is used by Lee and Moore ((13, pp. 274-280)).

CORELAP SOLUTION

A CORELAP solution to the given problem will be presented as if it were developed using a computer. No actual use was made of the CORELAP program, but this entire

Table 8

Unit Area Requirements and Interrelationship Requirements

Sq. Ft. Req'd. Area	Unit	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
50000	1	-	0	0	0	4	3	0	1	0	0	0	0	0	0	0	0	8
40000	2	0	-	2	0	0	0	3	2	2	0	0	4	0	0	2	0	15
60000	3	0	2	-	0	0	0	0	0	0	0	3	0	0	4	0	0	9
70000	4	0	0	0	-	1	0	0	0	0	2	0	0	1	0	0	0	4
60000	5	4	0	0	1	-	0	0	4	0	0	0	3	0	0	0	0	12
40000	6	3	0	0	0	0	-	1	0	0	3	0	0	4	0	0	0	11
50000	7	0	3	0	0	0	1	-	2	0	0	0	0	0	0	0	2	8
40000	8	1	2	0	0	4	0	2	-	0	0	0	1	0	0	0	0	10
50000	9	0	2	0	0	0	0	0	0	-	0	0	0	0	0	4	3	9
30000	10	0	0	0	2	0	3	0	0	0	-	2	0	0	0	0	0	7
40000	11	0	0	3	0	0	0	0	0	0	2	-	3	2	0	0	0	10
100000	12	0	4	0	0	3	0	0	1	0	0	3	-	0	0	0	0	11
40000	13	0	0	0	1	0	4	0	0	0	0	2	0	-	3	0	0	10
50000	14	0	0	4	0	0	0	0	0	0	0	0	0	3	-	0	0	7
110000	15	0	2	0	0	0	0	0	0	4	0	0	0	0	0	-	2	8
110000	16	0	0	0	0	0	0	2	0	3	0	0	0	0	0	2	-	7

Value	Closeness
4	Necessary
3	Very Important
2	Important
1	Ordinary
0	Unimportant

solution is based on the logic and programming presented by Major Robert C. Lee in his thesis ((12)). It is a path-oriented logical analysis which developes the layout through a series of building block crystals. These are constructed using comparative decision steps in selecting "a most related" activity and then placing next to it a facility which is "most related" to the two preceding activities to have been placed into the layout by the program. This program does not assert that its solution is optimal. The following flow diagram, Figure 19, is a synopsis of the actual diagram used by Major Lee. It was published by Lee and Moore ((13)). The following definitions are necessary to understand the flow diagram:

Victor: A department which has been selected to be placed in the layout during that iteration.

Winner: A victor from a previous iteration.

See Figure 19 on following page.

To determine the initial winner, CORELAP, using the matrix given in Table 8, produces a working matrix shown in Table 9, below. This matrix, called NARAY, is also used to determine the number of building squares per unit to be placed into the layout. This area requirement portion of the program has been simplified for the sake of clarity. No loss of generality occurs due to this simplification.

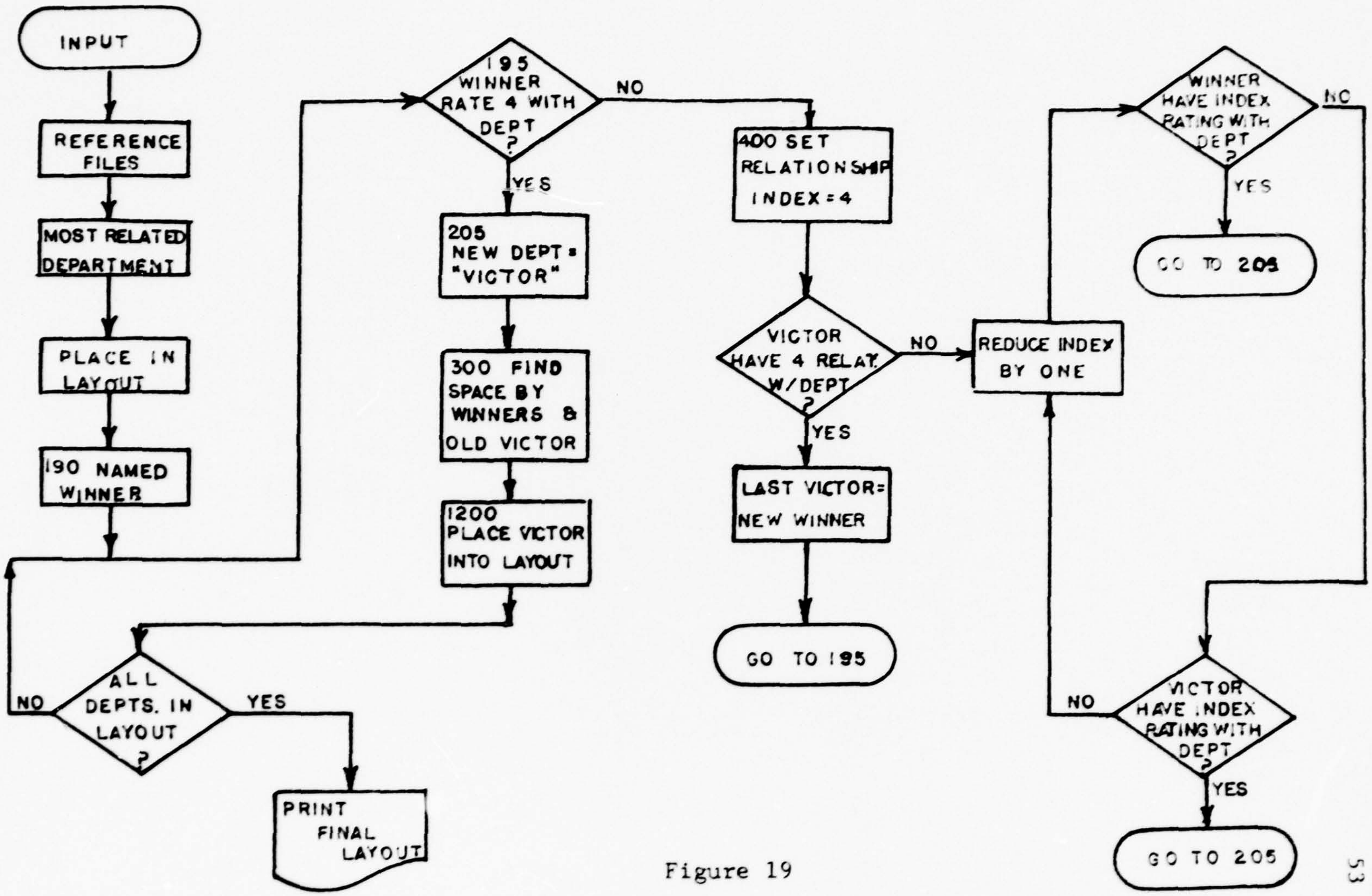


Figure 19

General Flow Diagram

Table 9

Ordered Relationship Matrix and Size Requirements, NARAY

Unit No.	Squares Required	Closeness Rating
2	4	15
5	6	12
12	10	11
6	4	11
8	4	10
11	4	10
13	4	10
3	6	9
9	5	9
15	11	8
1	5	8
7	5	8
16	11	7
14	5	7
10	3	7
4	7	4

Each square is 100 feet by 100 feet.

Using Table 9, CORELAP selects unit number 2 to be the first one to enter the layout. Figure 20 shows the printout following the first iteration. The program then searches the matrix represented in Table 8 to find another unit that has a closeness index of 4 with Unit 2: Unit 12 is selected and becomes the second unit to enter the layout as seen in Figure 21. The third iteration, not finding another index of 4 with Unit 2, looks for and does not find a 4 index with Unit 12. The index is lowered by one and Unit 7 is the first one encountered with an index of 3. It is selected and enters the layout on the third iteration printout shown in Figure 22. The process then continues selecting units to enter the layout based on the closeness rating between those not yet included and two of the units already in the layout. Figure 23 depicts the final layout for the complex. The outlines of the various units have been added.

In his thesis, Major Lee points out that the CORELAP solution, as illustrated in Figure 23, provides a good starting point for adjustment. The planners can adjust and shift with this layout as a guide.

Several of the more glaring discrepancies in Figure 23 are due to the fact that the top priority closeness index

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	2	0	0	0	0
0	0	0	0	2	2	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Figure 20
First Computer Printout

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	12	0	0	0
0	0	0	0	0	0	12	12	12	0
0	0	0	0	2	2	12	12	12	0
0	0	0	0	2	2	12	12	12	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Figure 21
Second Computer Printout

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	7	7	7	12	0	0	0
0	0	0	0	7	7	12	12	12	0
0	0	0	0	2	2	12	12	12	0
0	0	0	0	2	2	12	12	12	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Figure 22
Third Computer Printout

0	0	0	0	0	11	11	8	8	0
0	7	7	7	12	11	11	8	8	0
3	3	7	7	12	12	12	5	5	1
3	3	2	2	12	12	12	5	5	1
3	3	2	2	12	12	12	5	5	1
14	14	13	13	9	9	15	15	1	0
14	14	13	13	9	9	15	15	0	0
14	6	6	6	6	9	15	15	0	0
4	10	10	10	16	15	15	15	0	0
4	4	4	4	16	16	15	15	0	0
0	0	4	4	16	16	16	16	0	0
0	0	0	0	16	16	16	0	0	0

Figure 23

Final Computer Printout
CORELAP Solution Layout

overrides all others. It would require an extremely lengthy and complicated computer program to consider all the interrelations between the various units. Basically, the CORELAP program considers two relationships simultaneously.

GRAPH THEORY SOLUTION

The same problem will now be attacked with graph theory. Using the data given in the relationship chart in Table 8, a graph is constructed. The units to be positioned in the complex become the nodes of the graph and the relationships between the units become the edges. The graph of the given problem is shown in Figure 24 on the next page. The weight on the edges represent the closeness rating between the two unit nodes at either end of the edge. The

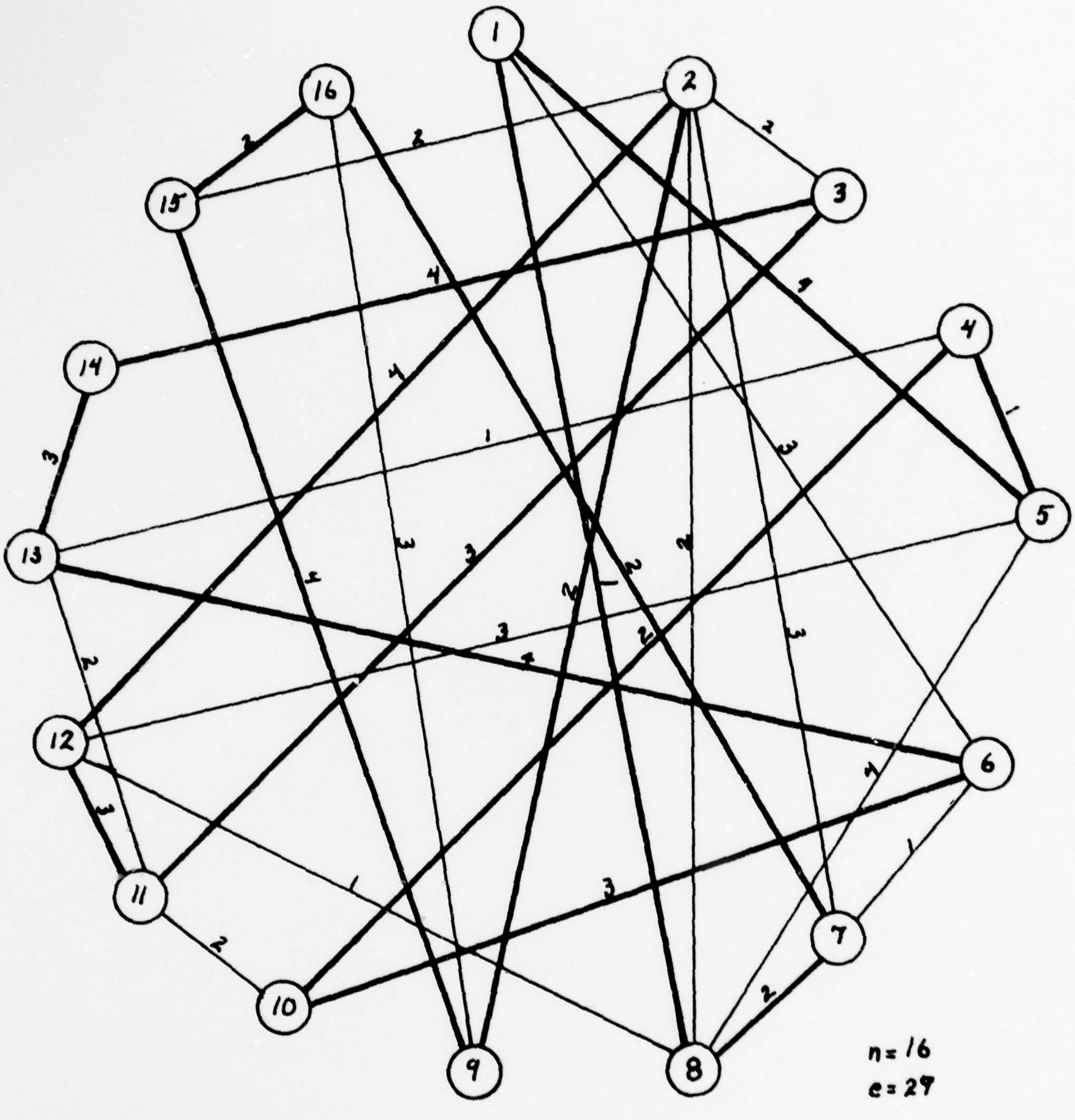


Figure 24
The Graph, G, of Logistic Complex
for LOGANIA

initial process of reducing the original graph to a planar graph is based on the article by Seppanen and Moore ((20, pp. B-242 to B-253)). Only the method of reducing a given graph will be demonstrated in the solution to the problem. The article discusses the background and more technical graph theoretic concepts dealing with the feasibility of solutions to general problems of this type. The method of reducing the given graph to a planar graph requires three steps, and the subsequent reduction of the planar graph to a proposed layout requires three additional steps.

Step One: Construct auxiliary graph G'

The graph G given in Figure 24 is obviously non-planar therefore, the first part of step one is to find a Hamiltonian cycle μ_H if one exists in G. A Hamiltonian cycle is one which visits all the nodes only one time. A Hamiltonian cycle in G in Figure 24 is shown in heavy edges and is the following cycle:

1-8-7-16-15-9-2-12-11-3-14-13-6-10-4-5-1.

Graph G is then redrawn with the Hamiltonian cycle as the exterior edges. The rearranged graph G is shown in Figure 25 on the following page. The second part of step one is to construct the auxiliary graph G' using as nodes the edges of $G \not\in \mu_H$. The edges of G' are drawn between nodes if the corresponding edges in the rearranged G are incompatible;

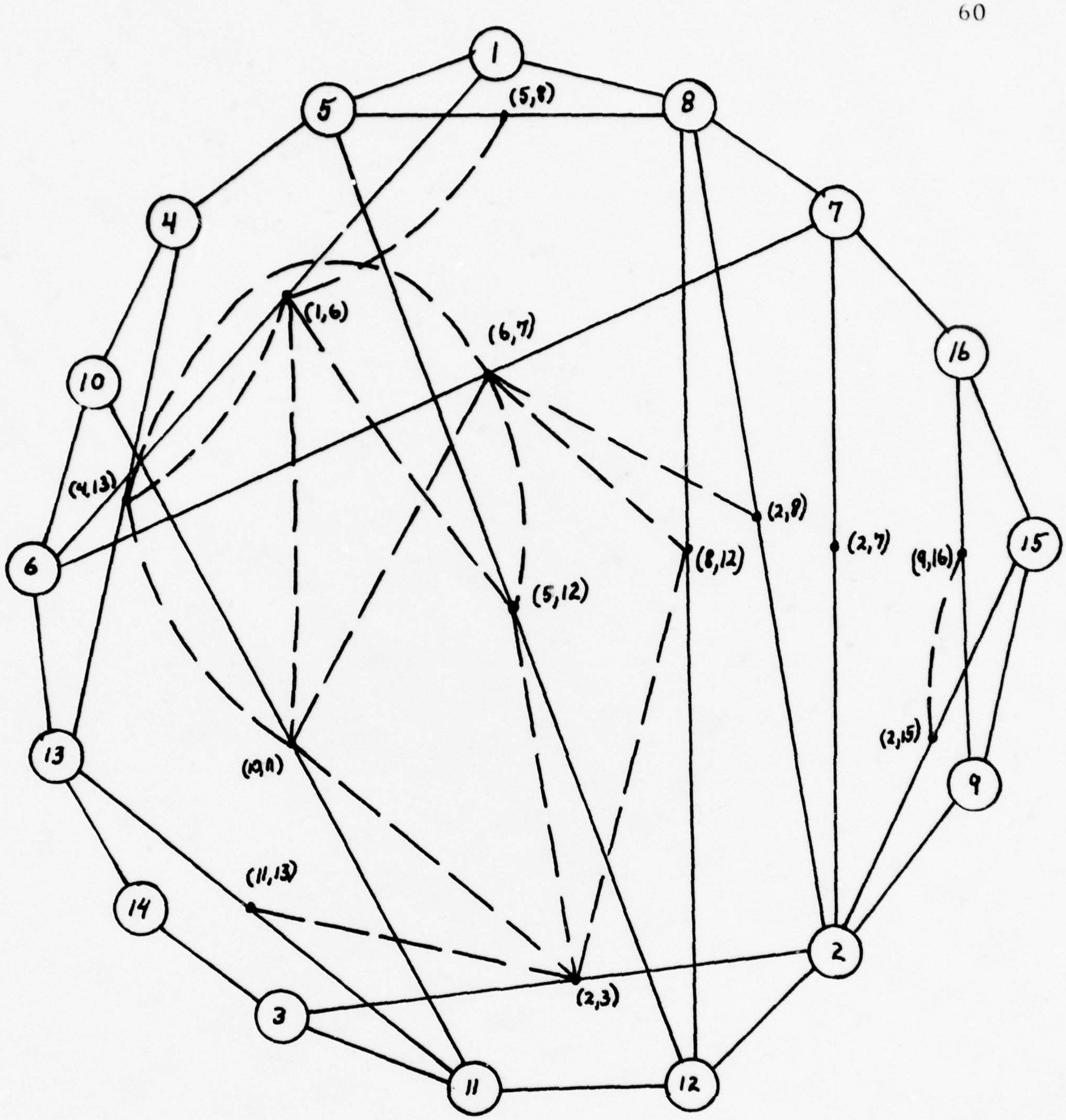


Figure 25
G Rearranged and G'

that is, if they intersect. The graph G' is represented in Figure 25 also. The dots on the edges which are labeled are the nodes and the dashed lines are the edges of G'.

Step Two: Rearrange G' and eliminate odd cycles

Figure 26 below shows G' after it has been rearranged.

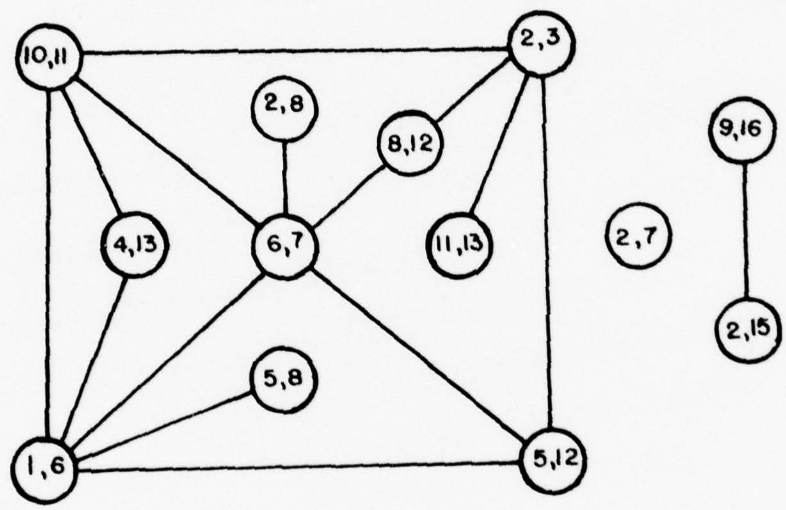


Figure 26

G' Rearranged

Note that the nodes of G' represent edges of G.

The second part of step two is the elimination of all odd cycles. The reason for this is explained in the theorem by König ((1, p. 10)). A cycle is odd if it contains an odd number of edges. Therefore, all such cycles can be eliminated in G' by deleting nodes 6,7 and 4,13.

Should a selection process in deciding which nodes to eliminate be necessary, the weight of the corresponding edges in G could be used in selecting the optimum. This produces the graph G" shown in Figure 27 below. This graph is now bichromatic; that is, it can be colored using only two colors as shown on the Figure.

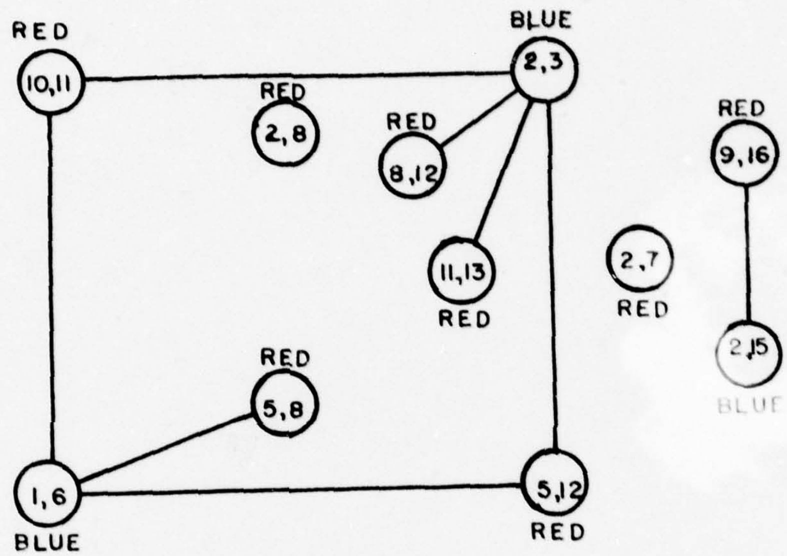


Figure 27

The Bichromatic Graph G"

Note that deletion of the 6,7 and 4,13 nodes of G' is equivalent to deleting those edges from G. The nodes remaining in G" now represent two groups of edges in G.

Step Three: Construct G_p (G minus deleted edges)

Using the two color labels from G", one color representing inside and the other outside, graph G_p is constructed

as shown in Figure 28. Note that G_p is a planar graph. The elimination of the odd cycles in G" made this change in G. Thus G has been transformed into a planar graph by the elimination of only two edges.

Step Four: Construct $\overline{G}_p$ (the dual)

The dual of G_p is also shown on Figure 28. The faces of G_p are lettered a through m and are connected by dashed lines. The dashed lines connecting the infinite face, a, to faces l,m,g,h,i and j have been combined for clarity.

Step Five: Redraw $\overline{G}_p$.

Since the faces of $\overline{G}_p$ now represent the nodes of G_p, that means that the faces of $\overline{G}_p$ can represent the units to be placed in the layout and the edges of $\overline{G}_p$ represent the boundaries between the units. As can be seen in Figure 29 all the adjacencies mentioned in the requirements except 6,7 and 4,13 are fulfilled.

Step Six: Deform G_p into layout

The final step requires that the redrawn $\overline{G}_p$ undergo an elastic deformation. The edges are stretched like rubber bands and the position of the nodes adjusted to conform to the shape of the units. The space requirements listed

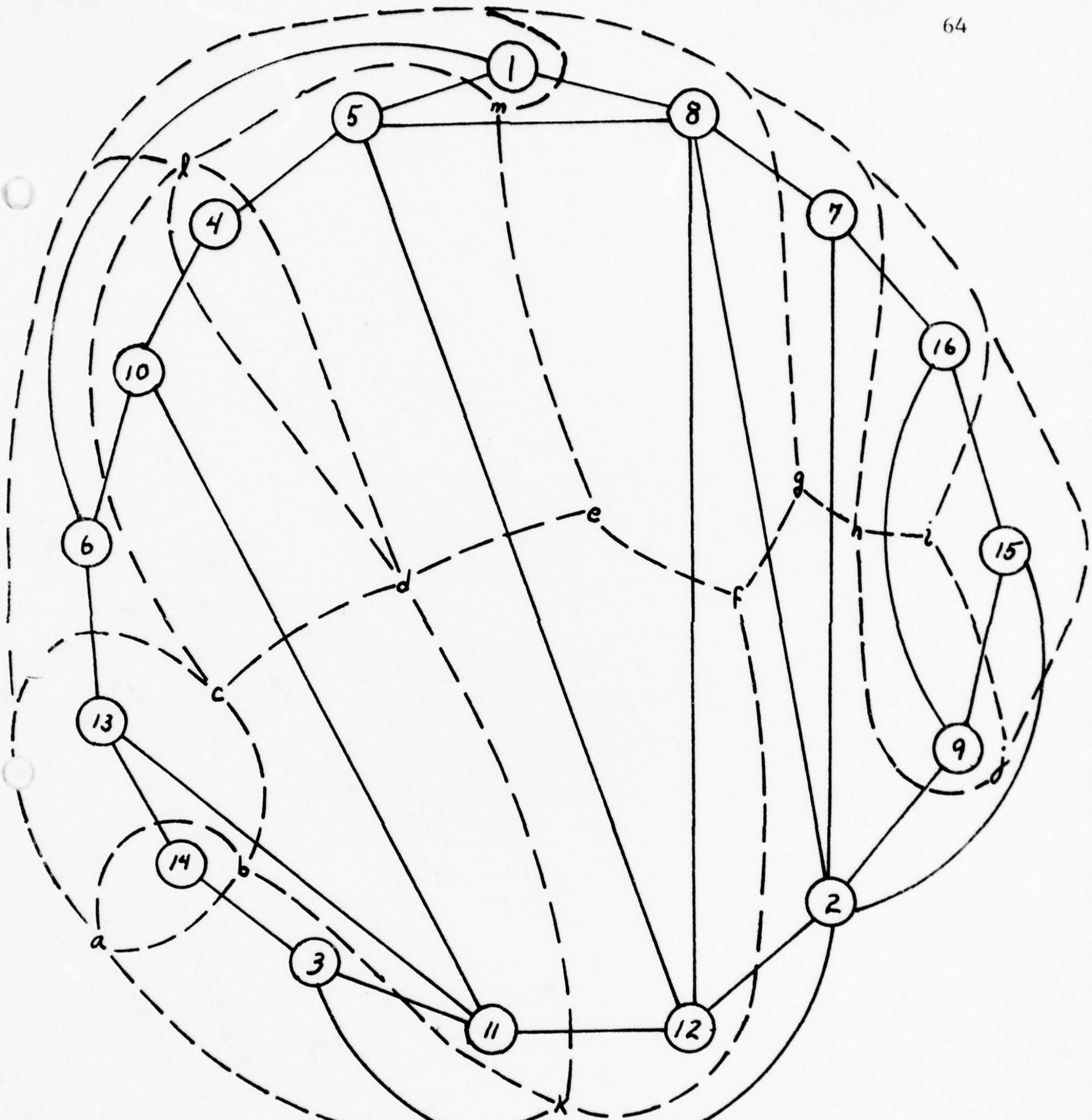


Figure 28

The Graph G_p and its Dual $\overline{G}_p$

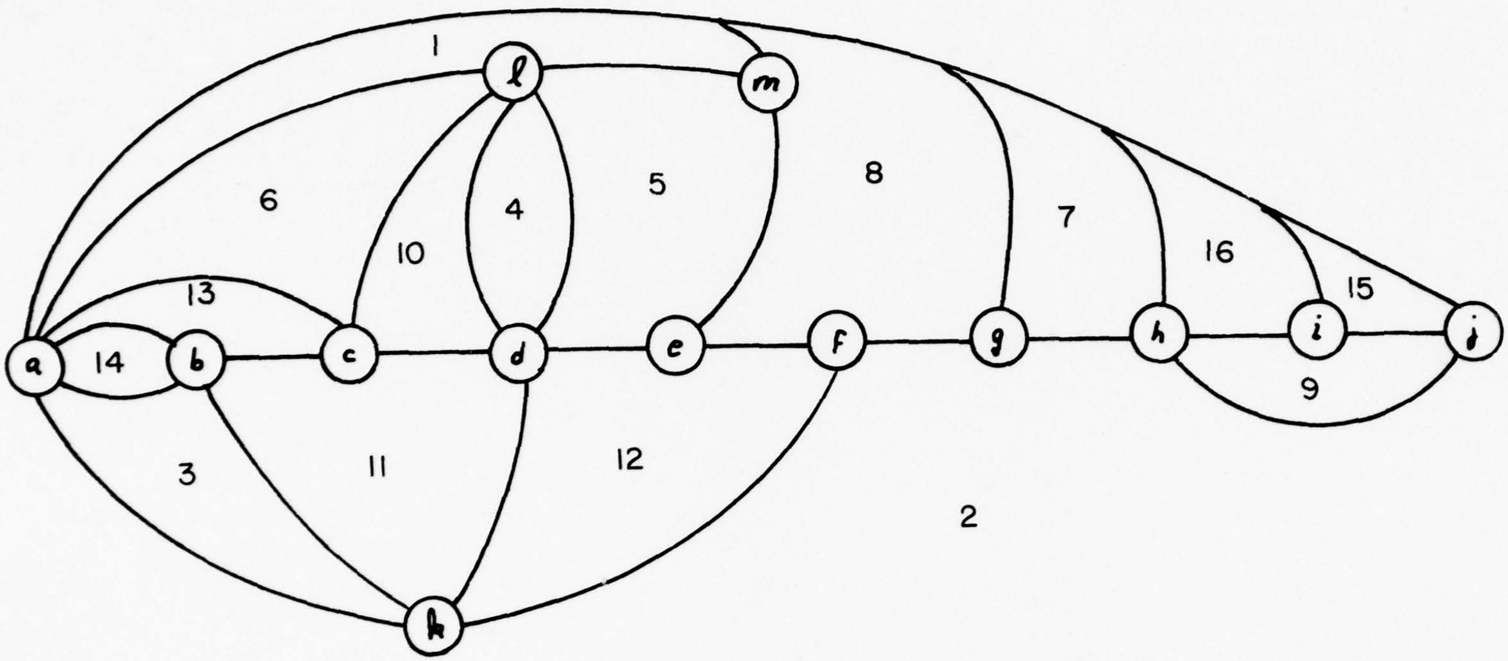


Figure 29

$\bar{G}_p$ Redrawn

initially in Table 8 are used to construct rectangular or square units. They are then positioned using $\bar{G}p$ as a guide. The result is shown in Figure 30 below. The Figure is drawn to scale and is constructed using square blocks representing 100 feet by 100 feet, similar to the CORELAP printout.

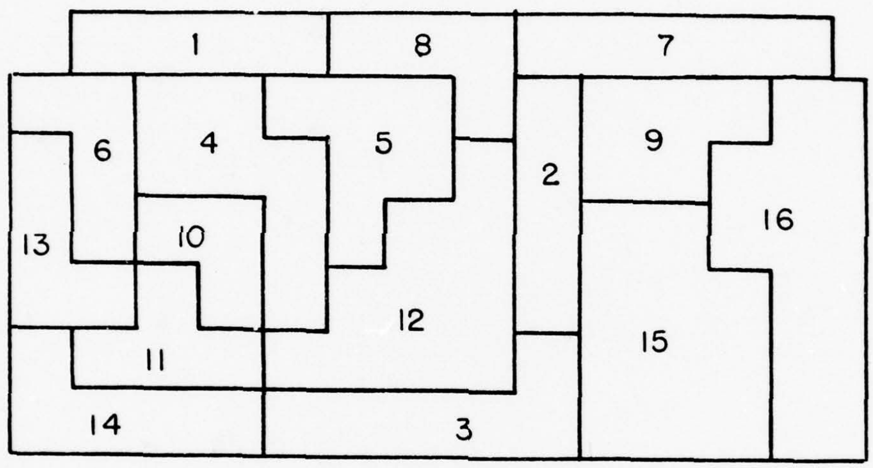


Figure 30

Solution Layout
by Graph Theory

Note that all of the closeness requirements, with the exception of the 6,7 and 4,13 have been met.

COMPARISON

In comparing the two methods of solving the layout problem, one must remember that final minor adjustments must

be made to both solutions. Peculiarities of the particular units must be considered. Additionally, roads and other features must be included in the final detailed plan.

It is significant, however, that in the CORELAP solution there were ten closeness requirements which were not met. The total value of those ten ratings is 19. In a graph theoretic solution only the two edges which were eliminated are missing. Their total closeness value is 2. The greatest value that these two ratings could have totaled is 8, while the least possible value of ten ratings is 10.

In the case of CORELAP, as mentioned before, only two closeness ratings are considered at any one time. The graph theory approach, presented here, considers all interrelationships between all units. This type of analysis is extremely complex when one considers the number of possible interrelationships. In this problem there were only twenty-nine, as can be seen in Figure 24. There were one hundred and twenty possible interrelationships. The total number of possible relationships in a general situation with n activities is $n(n - 1)/2$.

To use the CORELAP program, one must have a relatively fast computer and must have the program. There are other programs which attack this problem, but CORELAP is a new approach and Major Lee was acclaimed by the industrial

engineers at the seminar.

There are restrictions to graph theory, when analyzing this type problem. First, the planarity of the graph must be determined. Usually in this type layout problem that becomes quite obvious. Most are not planar. Seppanen and Moore ((20, p. B-246)) outline a method for testing for planarity. Second, a Hamiltonian cycle must be found. Again, this is usually not too difficult in a larger, more complex layout problem. There are several algorithms to locate Hamiltonian cycles. One is mentioned by Seppanen and Moore ((20, p. B-248)).

Finally, neither the graph theory approach nor the CORELAP solutions claims to be optimal. The fact that more than one Hamiltonian cycle may exist in the graph of the problem may require that more than one solution be investigated. An optimal selection technique for the removal of the nodes in a large scale G" graph has not yet been developed, although this thesis does consider the weighting of nodes. Therefore, both CORELAP and graph theory may develop into much better methods for solving this type problem. However, if one must solve such a problem without a computer or without having a program such as CORELAP available, graph theory provides a useful tool.

CHAPTER VI

TRANSPORTATION PROBLEM

This chapter will deal with the traditional transportation problem. This problem has been discussed in countless textbooks and articles. The general problem can be expressed in the following manner. There exists a number of sources $a_1, a_2, \cdots, a_n$ which produce and send a particular material over routes $x_1, x_2, \cdots, x_m$ to a number of recipients $b_1, b_2, \cdots, b_k$. A cost or work is associated with each route. The transportation problem consists of finding an optimum flow which minimizes the cost or work. There are numerous computer programs which will solve these type problems in an optimal manner. However, according to Gass ((8, p. 209)) the number of iterations and computer time required can be greatly reduced by an accurate first approximate solution. The object of this discussion is to compare two methods of obtaining such an approximation. Should a computer not be available to a U. S. Army planner faced with such a transportation problem, this chapter should provide an example of how such a problem can be

approached without an extensive background of complex mathematical procedures.

As in the previous chapters, the problem will first be introduced. Then the problem will be solved using the "Northwest Corner Method" which was introduced by Professor G. G. Danzig of California University ((8, p. 201)). The entire solution will not be carried to completion. As can be seen in the U. S. Army Command and General Staff Reference Book 20-5, Vol. II, Readings in Command Management ((23, pp. 17-11 to 17-17)), this procedure is extremely complicated. Therefore, Danzig's northwest corner technique as given by Gass ((8, p. 196)) will be used to obtain a first approximation. A graph theory technique developed in this research will then be used to obtain a solution to the same problem. Finally, a comparison of the two techniques will be made.

THE PROBLEM

The general transportation problem can be stated in many different ways. Even the mathematical statements of the problem can take many different forms. An example is given below.

Given two disjoint sets:

$$A = \left\{ a_i \,\middle|\, i = 1,2,\cdots,n \right\} \text{ and}$$

$$B = \left\{ b_j \mid j = 1,2,\cdots,m \right\}$$

with a set X $\left\{ x_{ij} \mid x_{ij} \text{ a flow between } a_i \text{ and } b_j \right\}$

two sets $C = \left\{ c_i \mid c_i \geq 0 \right\}$

$D = \left\{ d_j \mid d_j \geq 0 \right\}$

and a set $L = \left\{ l_{ij} \mid l_{ij} = \text{value associated with } x_{ij} \right\}$

and initially $a_i = c_i$ and $b_j = 0$

Find a flow

$$X = (x_{ij}) \quad i = 1,2,\cdots n \; ; \; j = 1,2,\cdots m$$

such that (1) $x_{ij} \geq 0$

(2) $a_i = 0$ and $d_j = b_j$

(3) $\sum_{ij} l_{ij} x_{ij}$ is a minimum

The problem used in this chapter is a reconstruction of a problem which has been presented to the cadets of the U. S. Military Academy. It was presented using different terms, but identical numbers. The cadets were only asked to formulate the mathematical statement of the problem. An optimal solution will be given in the comparison following the graph theory solution.

There are three logistic support bases I, II and III which support five different tactical areas 1 through 5. For a given period of time the amount of daily supplies in short tons available at each support base and required at each tactical area are shown in Table 10 below. The values

inside the matrix are the number of kilometers between the respective bases and areas. The transportation problem is to minimize the number of kilometers traveled in the daily delivery of the supplies.

Table 10

The Transportation Problem

Tactical Area / Support Base	1	2	3	4	5	Amount Available S.T.
I	55	30	40	50	40	40
II	35	30	100	45	60	20
III	40	60	95	35	30	40
Daily S.T. Requirements	25	10	20	30	15	100 (total)

Note that the right column represents the c_i and the bottom row d_j. The a_i are the bases I, II and III. The b_j are the tactical areas. The distances between the bases and the areas are the l_{ij}. The problem of minimizing the distance traveled daily equates to minimizing $\sum_{ij} l_{ij} x_{ij}$.

NORTHWEST CORNER METHOD

The following construction of an initial basic

feasible solution is based on the method given by Gass ((8, pp. 196-198)). First a tableau is constructed which represents all of the given variables of the problem. Using Table 10 the tableau is constructed, and is merely a reproduction as seen in Figure 31.

55	30	40	50	40	40	
35	30	100	45	60	20	
40	60	95	35	30	40	
25	10	20	30	15	0	- Total km.

Figure 31

Initial Tableau

The northwest corner of the tableau, representing x is first used to fulfill the requirements at area 1. Thus 25 of the 40 S.T. at base I is sent to area 1 at a distance of 55 km. The second tableau shown in Figure 32 below depicts the condition after the first step.

25x55 1375	30	40	50	40	15 ~~40~~	
-	30	100	45	60	20	
-	60	95	35	30	40	
0	10	20	30	15	1375	- Total km.

Figure 32

First Iteration

The northwest-corner rule now dictates that 10 of 15 S.T. remaining at base I is shipped to area 2. This results in

the following tableau:

25x55 1375	10x30 300	40	50	40	5
-	-	100	45	60	20
-	-	95	35	30	40
0	0	20	30	15	1675

Figure 33

Second Iteration

Continuing, the last 5 S.T. at base I is shipped to area 3 along with 15 S.T. from base II.

25x55 1375	10x30 300	5x40 200	-	-	0	
-	-	15x100 1500	45	60	5	
-	-	-	35	30	40	
0	0	0	30	15	3375	- Total km.

Figure 34

Third Iteration

The 5 S.T. from base II is shipped to area 4 along 25 S.T. from base III.

25x55 1375	10x30 300	5x40 200	-	-	0	
-	-	15x100 1500	5x45 225	-	0	
-	-	-	25x35 875	30	15	
0	0	0	0	15	4475	- Total km.

Figure 35

Fourth Iteration

Finally, the last shipment of 15 S.T. is made from base III to area 5 and the procedure is completed.

25x55 1375	10x30 300	5x40 200	-	-	1875	
-	-	15x100 1500	5x45 225	-	1725	
-	-	-	25x35 875	15x30 450	1325	
1375	300	1700	1100	450	4925	- Total km.

Figure 36

Fifth and Final Iteration

The number of total kilometers, 4925, is, of course, not a minimum. It can be illustrated by the following equation:

$$25x55 + 10x30 + 5x40 + 0x + 0x + 0x + 0x + 15x100 + 5x45 + 0x + 0x + 0x + 0x + 25x35 + 15x30 = 4925$$

or

$$1375 + 300 + 200 + 0 + 0 + 0 + 0 + 1500 + 225 + 0 + 0 + 0 + 0 + 875 + 450 = 4925.$$

Which means, in terms of the original formulation of the problem,

$$\sum_{ij} l_{ij} x_{ij} = 4925.$$

This then is the first feasible solution to this transportation problem based on the northwest-corner method.

GRAPH THEORY SOLUTION

This problem can be quickly organized into a graph. The edges in this case become directed, are called arcs and are represented by arrows. The nodes in a directed graph, or diagraph, are called vertices. The problem can initially be represented by the graph in Figure 37. The weight on the arcs in the Figure are of different natures. The values in parentheses on the arcs coming from vertex a, the source, are the amounts of materials available at the bases. The amounts, in parentheses, on the arcs going to b, the sink, are the amounts needed at the areas. Those amounts on the arcs between the bases and the areas are the distances involved. This graph can be simplified and can be constructed from the original data. If the source and the sink are considered imaginary and the values attributed to the bases and the areas directly the graph can be constructed as shown in Figure 38. The weights on the arcs are determined by multiplying the smaller of the two numbers at either end of the arc times the distance represented by the arc. For example, the arc from 1 to I represents a length of 55 km. and the smallest value on the vertices on either end is 25. Hence, 25 x 55 = 1375.

The first iteration in this graph procedure, as do all subsequent iterations, calls for circling the smallest

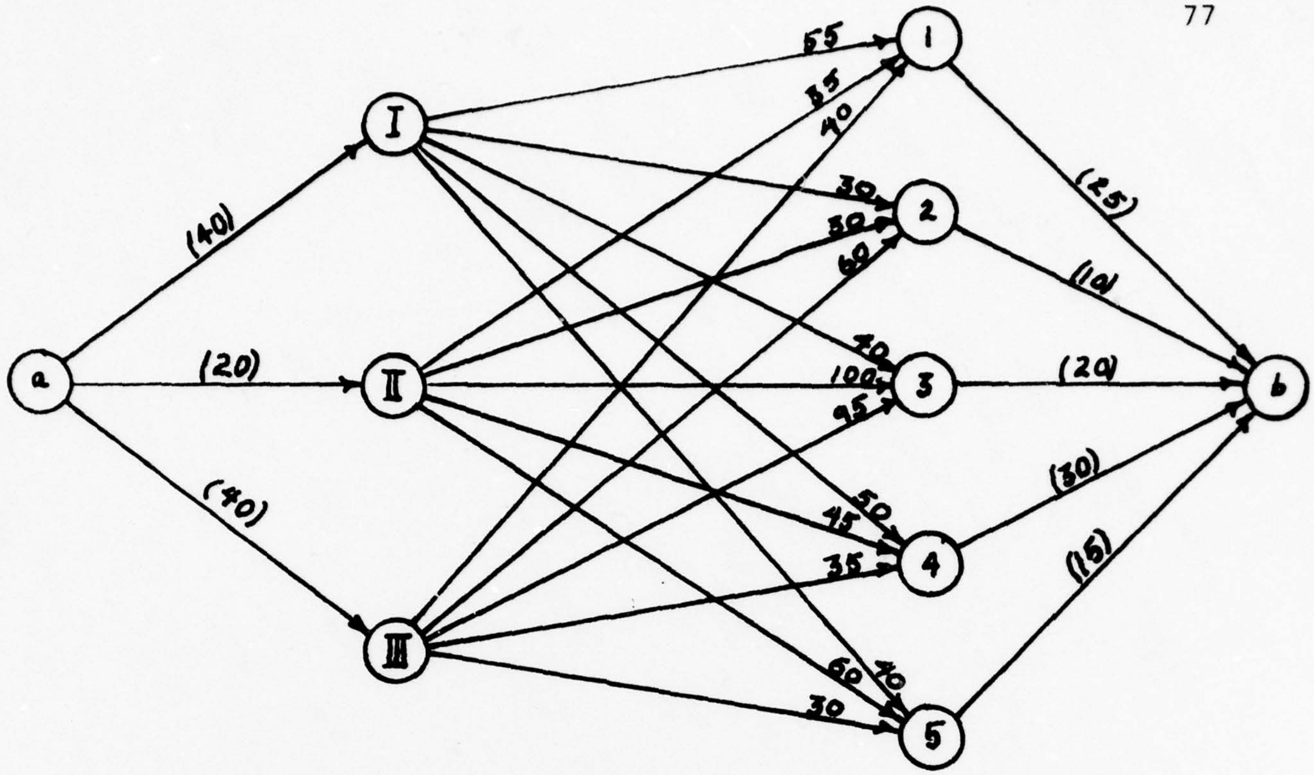


Figure 37

Graph of Transportation Problem

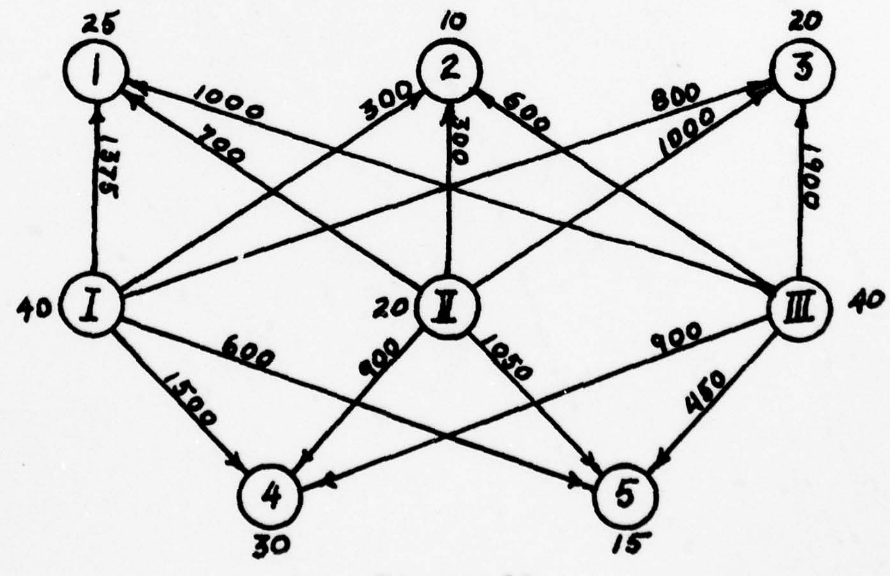
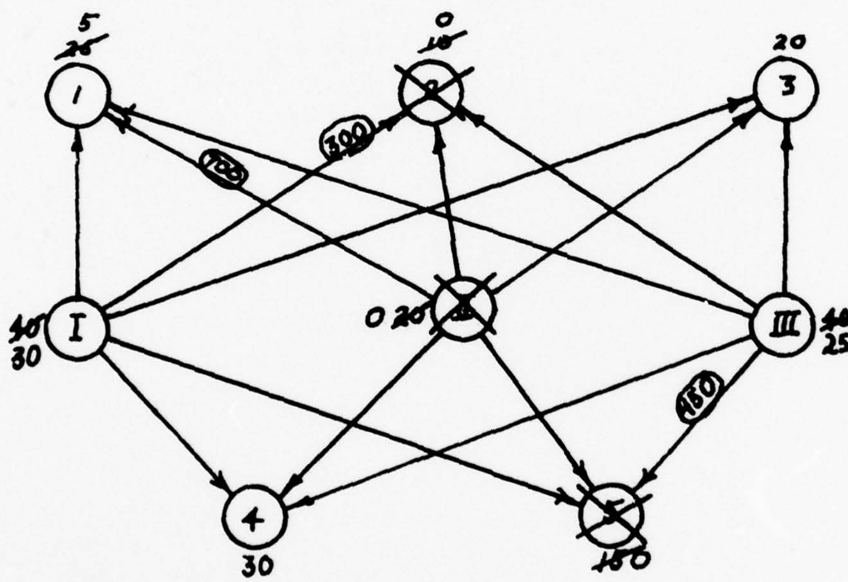


Figure 38

Simplified Graph for Transportation Problem

value on the arcs into the receiving vertices. Then using the indicated arcs, fill the receiving vertices with as much flow as those arcs allow. Use only arcs coming from separate vertices on any given iteration. Following these rules produces the resultant graph shown below. Those receiving vertices which are filled are zeroed out and crossed, as are those sending vertices which are empty. Note also that the values beside the vertices are changed dependent on the flow.



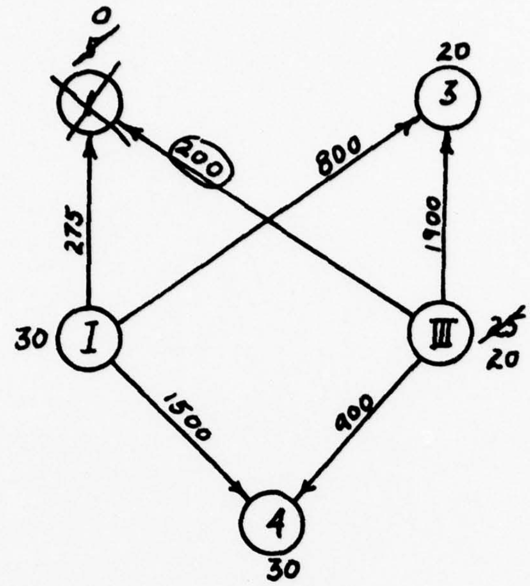
Total = 700+300+450 = 1450 km.

Figure 39

First Graph Iteration

The graph is then redrawn omitting those vertices which have been zeroed. The weights on the arcs are recomputed based on the new values at either end of the arcs. See Figure 40. The smallest value on the arcs is selected and circled.

That arc is used for the flow necessary to fill the requirements.



Total = 200 km.

Figure 40

Second Graph Iteration

The graph is redrawn with the non-zeroed vertices as shown in Figure 41. The smallest values on the two arcs as circled in the Figure are again selected and used.

The final graph, in Figure 42, shows the remaining vertices left for the final iteration.

By compiling all of the arcs used in this graphic solution and adding the total kilometers, the final graph theoretic solution is obtained and appears in Figure 43. Therefore, the initial graph theoretic feasible solution to the transportation problem is given below.

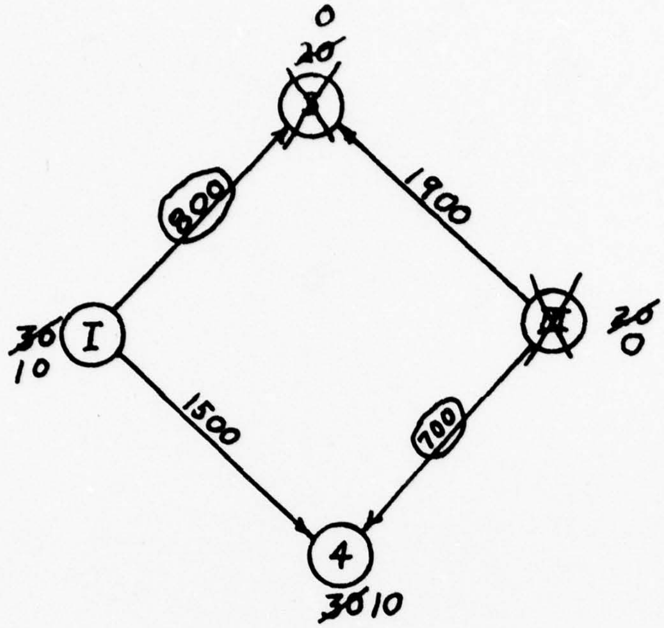


Figure 41

Third Graph Iteration

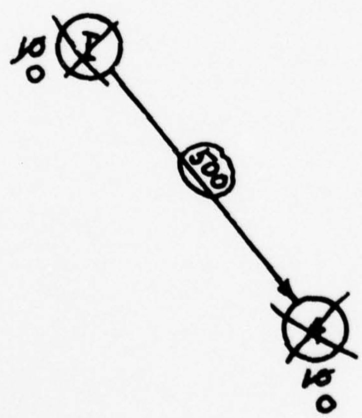


Figure 42

Fourth Graph Iteration

1450 + 200 + 1500 + 500 = 3650 km					

	10x30 300	20x40 800	10x50 500		1600
20x35 700					700
5x40 200			20x35 700	15x30 450	1350
900	300	800	1200	450	3650

Figure 43

Final Graph Solution

COMPARISON

To compare the two solutions to the problem of obtaining an initial feasible solution it is necessary to use the actual optimum solution. The solution to the given problem, obtained from a computer package transportation program is shown below.

	10x30 300	20x40 800		10x40 400	1500
20x35 700					700
5x40 200			30x35 1050	5x30 150	1400
900	300	800	1050	550	3600

Figure 44

Actual Minimum Solution

As can be seen, the graph theoretic approach developed in this thesis is only slightly above the actual minimum. The

reason for this is the fact that all the arcs in the graph are considered simultaneously. A maximum flow on each arc is considered at each iteration; therefore, it will not necessarily produce the absolute minimum. The graph theory method produced a closer first feasible solution than did the northwest-corner rule. Both required about the same degree of sophisticated knowledge of mathematics; namely, just arithmetic.

In analyzing the two methods, numerous actual transportation problems were investigated. On every one the graph theory method produced a solution which was closer to the minimal solution than did the northwest-corner rule. The greatest deviation from the minimum observed using the graph theory method was 5%. In the example given here the deviation is only 1%. This is considerably better than the 36% of the northwest-corner rule.

CHAPTER VII

FINDINGS ON GRAPH THEORY

This final chapter of the thesis will first present a summary of the entire study. The hypothesis will be restated and the findings of the three comparative problems will be discussed. Following the summary, conclusions will be enumerated and discussed. Finally, recommendations for applications and for further study will be presented.

SUMMARY

Although graph theory has been in existence since 1736, it has just recently been applied to solving problems in many diverse fields. It was originally thought to be applicable only to puzzles and games. The abstract theory behind the subject has been developed recently, to the point that it is a field of mathematics unto itself. In addition to the abstract concepts, the list of practical applications of graph theory has been expanding.

Graph theory is a mathematical discipline which deals with two sets and the relationship rules which link

the two sets. One set is made up of nodes or vertices. The other set is made up of edges or arcs. The rules for combining the two sets are dependent on the type situation which is being modeled by the graph. The complete generality of graph theory and the flexibility inherent in the defining of the rules, make it readily available as a mathematical model.

Some applications of graph theory are already well known. PERT is but one example. However, it is not as widely used as it could be, particularly in the Army. The numerous articles about graph theory which appear in technical publications are habitually extremely technical in nature. The fact that graph theory can be understood and used without an extensive mathematical background is not well known. Therefore, the following hypothesis was proposed: If graph theory is compared to the methods currently being used by the U. S. Army in solving several managerial problems, it will become apparent that graph theory is superior.

In the scheduling problem, the final comparison showed that the choice was between an analytical approach and a semi-analytical one. Most problems of the type presented occur at locations where a computer is not readily available. A feasible solution is usually acceptable,

although an optimum one would be better. The graph theory solution requires no more mathematical background than the method now being used. The same equipment, pencil and paper, is required for both methods of solution. Graph theory guarantees a systematic approach toward an optimal solution, while the trial and error method would require an excessive number of trials to guarantee an optimal solution. Both methods are adaptable to computer programs. Since the graph theory approach does not search the entire set of possible solutions but proceeds directly toward the optimal, it will provide a more economical program.

The layout problem presents a challenge to any method of solution. The number of possible arrangements of the activities within a facility that is being planned increases extremely rapidly as the number of activities increase. Any procedure which considers all of the inter-activity relationships is necessarily complicated. If a procedure considers them one or two at a time, it quickly becomes unmanageably long. In comparing graph theory with CORELAP, it became apparent that CORELAP was a more systematic approach. However being more systematic forced it to consider only a few of the inter-activity relationships. Graph Theory, on the other hand, was able to consider all of the prescribed relationships. It did become necessary for

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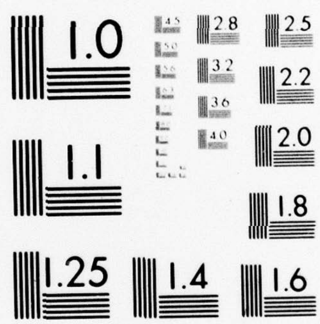

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

the graph theory algorithm to discard two of the relationships. After they were discarded, the layout became a problem of physically arranging the various sizes and shapes according to the final graph. This procedure was much less analytical than CORELAP.

The general transportation problem has been widely publicized and there are numerous computer package-programs which solve the various types. Many times when a transportation problem arises in an Army situation there is no computer available. Since both the northwest-corner rule and the illustrated graph theory method produce a feasible solution, either one could be used as a first approximation. Both methods require the same amount of mathematical background and equipment. Without resorting to one of the more complicated computational methods, either of these two methods could be used. They can be used for implementation or for further refining. If a computer is available, a first feasible solution will reduce the amount of computer time used by reducing the number of iterations. The closer the first solution is to the final minimal solution, the fewer the iterations and the shorter the computer time. In the example problem and in sixteen other representative transportation problems, the graph theory approach produced a closer first feasible solution than did the northwest-

corner rule. There is no mathematical proof that this will always be the case. In fact it may be possible for the northwest-corner rule to produce a closer solution.

CONCLUSIONS

Basic graph theory is not a complicated mathematical discipline. A graph model of a real world situation can be constructed without reference to, or detailed knowledge of, the more abstract theorems. A few basic definitions and rules are enough background for one to be able to use graphs to solve problems.

The algorithm for using graph theory to solve the type scheduling problem illustrated in this thesis is superior to the trial and error approach normally used by U. S. Army staffs. It requires the same or less equipment. It requires no more mathematical background. It also produces an optimal solution as its initial solution. The manual solution illustrated in this thesis would be difficult to use on a large problem. However the algorithm lends itself to the decision statements of computer programming. This is illustrated by the flow chart given in Chapter IV. On either large or small scheduling problems, graph theory provides a better method.

In dealing with the complexities of the layout

problem, the graph theory approach is able to consider many more of the inter-activity relationships than is CORELAP. However, CORELAP is much more systematic in introducing activities into the plan for the facility. Graph theory must rely on a manual analysis of the final graph. A comparison of the two methods shows that each has advantages over the other in particular areas. CORELAP can handle larger layouts but it does require a computer and a particular program. Graph theory, on the other hand, is able to realize more of the closeness relationships.

There are no provable advantages or disadvantages to using either graph theory or the northwest-corner rule in finding the first feasible solution to a transportation problem. The illustrative problem in Chapter VI demonstrated the fact that the graph theory approach can produce a closer first feasible solution. This may not always be true. Neither graph theory nor the northwest-corner rule require much time or background. Both methods can be used and their solutions compared prior to deciding on the best first feasible solution.

RECOMMENDATIONS

Graph theory could be used to a greater degree by U. S. Army managers in the solutions to problems of the type

illustrated in this thesis. Additionally, other similar problems which face the commanders and staffs at every level of the Army should be considered for possible solution by graph theory. There is sufficient research being conducted at the more abstract levels of graph theory. It is at the working level that graph theory should be more fully exploited. A resistance to use what appears to be complicated, mathematical techniques needs to be overcome. This is probably true of other analytical techniques also. However, in the case of graph theory, it has been shown in this thesis that it could be used and that it is superior to techniques now being used.

There are numerous other areas in which graph theory has begun to play a more important role. The military, and particularly the U. S. Army, could investigate, at the working level, the applications of graph theory in such areas as role theory, group structure, organization structure, and network flow as it pertains to the transport of supplies in Army rear areas.

In conjunction with the problems presented in this paper there are several areas where further study seems justified. The computer program for the scheduling problem could be developed and put to use. The bounds and limitations of the various types of scheduling problems should be

investigated. The same type bounds and limitations on the size and shapes of the layout problem should be studied. On the other hand, the idea of developing a computer program for the graph approach to the transportation problem does not seem to be worthwhile. The numberous existing programs seem sufficiently varied to handle the different cases which arise.

Graph theory is being used extensively by system analysts and mathematicians. It is being used in numerous diversified fields. Its uses should be evaluated by the working levels of the U. S. Army.

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